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An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain

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In this article, the controllability results of the non-dense Hilfer neutral fractional derivative (HNFD) are presented. The results are acknowledged using semigroup theory, fractional calculus, Banach contraction principle, and Mönch technique. Moreover, a numerical analysis is given to enhance our model.

1. Introduction

The generalization of traditional calculus to arbitrary order is fractional calculus; It has attracted several researchers with great potential in the current scenario because it is reliable and growing, employing both theoretical and applied concepts. Valuable tools investigate the hereditary property and memory description of various materials and processes from fractional calculus. The fractional derivatives were developed in the past epoch by Riemann–Liouville (R-L), Grünwald–Letnikov, Riesz, Erdélyi–Kober, Caputo, Hadamard, Hilfer, and others. In recent years, fractional differential equations have been considered as a beautiful, rich domain to be studied because of its applications in life sciences and to engineering, as is witnessed by blossoming literature. Several researchers expressed the natural derivatives of arbitrary order characterized by Riemann–Liouville and Caputo's sense. One can find the results [5,8–10,15,16,19,20,29,34,37,43–47] and monographs [1,13,18,21,24,26,48].

Recently, generalizations of both Caputo and R-L derivatives are introduced and reflected on equations of probability or mathematical physics. The same was achieved with Hilfer definition proposed by Hilfer [13,14]. Shortly, it behaves as interpolator between Caputo and R-L derivative [3,11,17,30–32,35,36,38–41]. Hilfer parameter produces many types of stationary states and gives more degrees of freedom related to an initial condition. It reacts to theoretical simulation in glass-forming materials. To solve generalized fractional systems, Hilfer et al. [14] introduced applied operational calculus. Besides, Gu et al. [11], Furati et al. [7], investigated the nonexistence, existence, and stability sequels of nonlinear problems with Hilfer derivative.

Control theory generally deals with dynamic system behavior and becomes one of the essential tools in the method of mathematical control. Controllability defines the control system in terms of qualitative property and plays a significant role in the theory of control. Controllability deals with problems on optimal control, pole assignment, stability employing the corresponding system is

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controllable. It is a tool used to drive the system from its arbitrary initial to the final state. The contribution of controllability by several researchers may be referred to [4,6,20,22,23,33,34,44,46], and references. The above articles refer to the investigation of controllability on C_0-semigroup with a dense operator that trivially meets Hille condition. To overcome real-life situations, one may go with non-dense operator, as suggested in Prato and Sinestrari [27]. On the other hand, optimal control of differential equations and inclusions with integer order is of interest for space technology and aviation. It also plays an important role in robotics, power plants, control of chemical processes and movement sequence of sports. Practically, optimization process can no longer be adequately modelled by integer order differential equations; instead differential equations of fractional order are employed for their description. For instance, the memory and hereditary properties of blood flow, electrical circuits, bio-mechanics, signals can be well predicted and described by some fractional differential equations. One can refer to the results in Bahaa [2], Harrat et al. [12], Pan et al. [25], Qin et al. [28].

In the year 2016, Yang and Wang [42] discussed the approximate controllability of Hilfer nonlocal differential inclusions of fractional order. Continuation of this in 2018, Du et al. [4] published an article regarding the controllability of nonlocal Hilfer fractional inclusion. In 2019, Vikram Singh [33] derived some results on the controllability of non-dense Hilfer equation of fractional order.

As per our vast search, there is no article found related to the investigation on controllability of non-dense HNFD which attracts us to make a study on the above-said title and followed by the problem as:

$$\mathcal{D}^{\alpha,\beta}_{0+}[\mathfrak{z}(\theta)-\mathcal{P}(\theta,\mathfrak{z}(\theta))] = A\mathfrak{z}(\theta)+Bu(\theta)+h(\theta,\mathfrak{z}(\theta)), \tag{1.1}$$

$$I^{(1-\alpha)(1-\beta)}_{0+}\mathfrak{z}(0)=\mathfrak{z}_0+\phi(\mathfrak{z}), \qquad \theta\in\mathcal{I}=[0,a]. \tag{1.2}$$

$\mathcal{D}^{\alpha,\beta}_{0+}$ denotes the derivative of fractional order in Hilfer sense with $\alpha\in(0,1)$, $\beta\in(0,1]$ as order and type respectively and $\vartheta=\alpha+\beta-\alpha\beta$. Here $A:\mathcal{D}_A\subset\mathcal{Z}\to\mathcal{Z}$, the non-densely closed linear operator, i.e. if we assume the conditions of Hille–Yosida with the density exception, $\mathcal{D}_A\in\mathcal{Z}$ where $\mathcal{D}_A,\mathcal{Z}$ represents the domain of A and Banach space respectively. Also the appropriate functions $\mathcal{P},h$ are defined as $\mathcal{P}:[0,a]\times\mathcal{Z}\to\mathcal{D}_A\subset\mathcal{Z}$ and $h:([0,a]\times\mathcal{Z})\to\mathcal{D}_A\subset\mathcal{Z}$. Also we consider the bounded linear operator $B:U\to\mathcal{Z}$ and the control function $u(\cdot)$ with the Banach space $L^2[\mathcal{I},U]$ of admissible control functions.

This article is outlined as: 2nd section introduces some notations and preliminary facts of semigroup theory, Mönch fixed point technique, fractional calculus and formulation of integral solution. In 3rd section, the uniqueness and controllability of integral solutions for (1.1) and (1.2) are established. 4th section refers to the existence of optimal control of our system. As a final part, in 5th section, a numerical analysis is given to compare the results with graphs.

2. Preparatory discussions

Let $\mathcal{C}(\mathcal{I},\mathcal{Z})$ be the space of continuous functions $\mathfrak{z}(\theta)$ defined on $\mathcal{I}=[0,a]$ provided with $||\mathfrak{z}||=\sup_{\theta\in\mathcal{I}}||\mathfrak{z}(\theta)||$.

$\mathcal{C}_{1-\vartheta}(\mathcal{I},\mathcal{Z})=\{\mathfrak{z}:\mathcal{Z}\to\mathcal{Z}$ such that $\theta^{1-\vartheta}\mathfrak{z}(\theta)\in\mathcal{C}(\mathcal{I},\mathcal{Z})\}$, a Banach space w.r.t. the norm $||\mathfrak{z}||_{\mathcal{C}_{1-\vartheta}}=\sup_{0\le\theta\le a}|\theta^{1-\vartheta}\mathfrak{z}(\theta)|$.

Here the basic definitions of Caputo and R-L derivatives are recalled:

$${}^{R-L}\mathcal{D}^{p}_{0+}z(\theta)=\frac{d^n}{d\theta^n}(z(\theta)*q_{n-p}(\theta)), \tag{2.1}$$

$${}^{C}\mathcal{D}^{p}_{0+}z(\theta)=\frac{d^n}{d\theta^n}z(\theta)*q_{n-p}(\theta), \qquad n-1<p<n, \tag{2.2}$$

where $z\in\mathcal{C}(\mathcal{I},\mathcal{Z})$ and $*$ denotes convolution of two functions.

Definition 2.1 (see [13]). For $\alpha\in(n-1,n)$, $n\in N$; $\beta\in(0,1]$, we define the HFD as

$$\mathcal{D}^{\alpha,\beta}_{0+}h(\theta)=\mathcal{I}^{\alpha(n-\beta)}_{0+}\frac{d}{d\theta}\mathcal{I}^{(1-\alpha)(n-\beta)}_{0+}h(\theta)=\mathcal{I}^{\alpha(n-\beta)}_{0+}\mathcal{D}^{\beta+\alpha n-\beta\alpha}_{0+}h(\theta)$$

where $\mathcal{I}^{\alpha(n-\beta)}_{0+}$ is R-L integral and $\mathcal{D}^{\beta+\alpha n-\beta\alpha}_{0+}$ is R-L derivative.

Lemma 2.2 (see [7]). *If $h\in\mathcal{C}^{\vartheta}_{1-\vartheta}[r_1,r_2]$ is such that $\mathcal{D}^{\vartheta}_{0+}h\in\mathcal{C}_{1-\vartheta}[r_1,r_2]$ then*

$$\mathcal{I}^{\vartheta}_{0+}\mathcal{D}^{\vartheta}_{0+}h=\mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha,\beta}_{0+} \text{ and } \mathcal{D}^{\vartheta}_{0+}\mathcal{I}^{\vartheta}_{0+}h=\mathcal{D}^{\beta(1-\alpha)}_{0+}h,$$

where $\alpha\in(0,1)$, $\beta\in(0,1]$ and $\vartheta=\alpha+\beta-\alpha\beta$.

Lemma 2.3 (see [7]). *If $h\in\mathcal{C}_{1-\vartheta}[r_1,r_2]$ and $\mathcal{I}^{1-\vartheta}_{0+}h\in\mathcal{C}^{1}_{\vartheta}[r_1,r_2]$ then,*

$$\mathcal{I}^{\vartheta}_{0+}\mathcal{D}^{\vartheta}_{0+}h(\theta)=h(\theta)-\frac{\mathcal{I}^{1-\vartheta}_{0+}h(r_1)}{\Gamma(\vartheta)}(\theta-r_1)^{\vartheta-1}, \quad \forall\theta\in(r_1,r_2],$$

where $\alpha\in(0,1)$, $\vartheta\in[0,1)$.

Remark 2.4 (see [11]).

(i) For $\beta=0$, $\alpha\in(0,1)$, $\mathcal{D}^{\alpha,0}_{0+}$ corresponds to classical R-L derivative: $\mathcal{D}^{\alpha,0}_{0+}h(\theta)=\frac{d}{d\theta}\mathcal{I}^{1-\alpha}_{0+}h(\theta)={}^{R-L}\mathcal{D}^{\alpha}_{0+}h(\theta)$.
(ii) If $\alpha\in(0,1)$, $\beta=1$, $\mathcal{D}^{\alpha,1}_{0+}$ corresponds to classical Caputo derivative: $\mathcal{D}^{\alpha,1}_{0+}h(\theta)=\mathcal{I}^{1-\alpha}_{0+}\frac{d}{d\theta}h(\theta)={}^{C}\mathcal{D}^{\alpha}_{0+}h(\theta)$.

Lemma 2.5 (see [10]). *We define $\kappa(\Omega)=\inf\{\epsilon>0,\ \Omega$ has finite $\epsilon-$net in $\mathcal{Z}\}$, the Hausdorff noncompact measure which satisfies:*

(1) *$\kappa(\Omega_1)\le\kappa(\Omega_2)$, for all bounded subsets Ω_1,Ω_2 of $\mathcal{Z}$ provided $\Omega_1\subseteq\Omega_2$;*
(2) *$\kappa(\Omega)=0$ if and only if Ω is relatively compact in $\mathcal{Z}$;*
(3) *for every $y\in\mathcal{Z}$, $\kappa(\{y\}\cup\Omega)=\kappa(\Omega)$, where $\Omega\subseteq\mathcal{Z}$ is nonempty;*
(4) *$\kappa(\Omega_1+\Omega_2)\le\kappa(\Omega_1)+\kappa(\Omega_2)$, where $\Omega_1+\Omega_2=\{y_1+y_2;y_1\in\Omega_1,y_2\in\Omega_2\}$;*
(5) *for any $\lambda\in R$, $\kappa(\lambda\Omega)\le|\lambda|\kappa(\Omega)$;*
(6) *$\kappa(\Omega_1\cup\Omega_2)\le\max\{\kappa(\Omega_1),\kappa(\Omega_2)\}$.*

Proposition 2.6. *Let $A_0\subset A$ generate a strongly continuous semigroup $\{\Re(\theta)\}_{\theta\ge0}$ on $\mathcal{Z}_0$ where $\mathcal{Z}_0=\overline{\mathcal{D}_A}$ satisfies $A_0y=Ay$.*

Lemma 2.7 (See [16]). *Let $\mathcal{I}$ be the set $[0,a]$, $\{z_n\}_{n=1}^{\infty}$ be a Bochner's sequence from $\mathcal{I}$ to $\mathcal{Z}$ satisfying $|z_n(\theta)|\le\widetilde{m}(\theta)$, $\theta\in\mathcal{I}$ with $n\ge1$, as $\widetilde{m}\in L(\mathcal{I},R^+)$. Moreover, the function $G(\theta)=\kappa\left(\{z_n(\theta)\}_{n=1}^{\infty}\right)$ in $L(\mathcal{I},R^+)$ fulfills*

$$\kappa\left(\left\{\int_0^{\theta}z_n(s)ds:n\ge1\right\}\right)\le2\int_0^{\theta}G(s)ds.$$

Consider $\mathcal{Z}_0=\overline{\mathcal{D}_A}$ and let A_0 be the characteristic element of A in $\mathcal{Z}_0$ defined as

$$\mathcal{D}_{A_0}=\{y\in\mathcal{D}_A:Ay\in\mathcal{Z}_0\},\ A_0y=Ay.$$

With reference [10], we introduced some assumptions for further analysis.

(H1) For couple of constants $k\in\mathbb{R},\mathcal{M}_0$ satisfying $(k,+\infty)\subseteq\rho(A)$, for each $n\ge1$ and $\lambda>k$,

$$\|(\lambda I-A)^{-n}\|_{L(\mathcal{Z})}\le\frac{\mathcal{M}_0}{\sup(\lambda-k)^n}.$$

(H2) There exists a constant $\mathcal{M}_1 > 1$ such that $\sup_{\theta\in[0,+\infty]} |\Re(\theta)| < \mathcal{M}_1$. i.e. $\{\Re(\theta)\}_{\theta>0}$ is bounded and uniformly continuous.

Now, for $\theta \geq 0$ we define,

$$T_\alpha(\theta) = \alpha \int_0^\infty \nu\psi_\chi(\nu)\Re(\theta^\alpha \nu)d\nu, \qquad P_\alpha(\theta) = \theta^{\alpha-1}T_\alpha(\theta),$$
$$S_{\alpha,\beta}(\theta) = \mathcal{I}_{0+}^{\beta(1-\alpha)}P_\alpha(\theta).$$

For $\nu \in (0,\infty)$,

$$\psi_\chi(\nu) = \frac{1}{\chi}\nu^{(-1-\frac{1}{\chi})}W_\chi(\nu^{-\frac{1}{\chi}}) \geq 0,$$
$$W_\chi(\nu) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\nu^{-k\chi-1}\frac{\Gamma(k\chi+1)}{k!}\sin(k\pi\chi).$$

Also, ψ_χ refers to probability density function on $(0,\infty)$.

Lemma 2.8. *(see [16]). By (H2),*

(i) $S_{\alpha,\beta}(\theta)$, $P_\alpha(\theta)$ *satisfy*

$$|S_{\alpha,\beta}(\theta)| \leq \frac{\mathcal{M}\theta^{\vartheta-1}}{\Gamma(\vartheta)} \quad \text{and} \quad |P_\alpha(\theta)| \leq \frac{\mathcal{M}\theta^{\alpha-1}}{\Gamma(\alpha)}, \ \theta > 0.$$

(ii) *For* $\theta \geq 0$, $T_\alpha(\theta)$ *is uniformly continuous.*

(iii) *For* $\mathfrak{z} \in \mathcal{Z}_0$, $0 < \theta_1 < \theta_2 \leq a$, $\{S_{\alpha,\beta}(\theta)\}_{\theta\geq 0}$ *and* $\{P_\alpha(\theta)\}_{\theta\geq 0}$ *satisfy*

$$|S_{\alpha,\beta}(\theta_1)\mathfrak{z} - S_{\alpha,\beta}(\theta_2)\mathfrak{z}| \to 0 \text{ and } |P_\alpha(\theta_1)\mathfrak{z} - P_\alpha(\theta_2)\mathfrak{z}| \to 0 \text{ as } \theta_2 \to \theta_1.$$

Lemma 2.9. *(see [7]). For $\theta \in \mathcal{I}$, our model (1.1) and (1.2) reduces as,*

$$\mathfrak{z}(\theta) = \frac{[\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))]}{\Gamma(\vartheta)}\theta^{\vartheta-1} + \mathcal{P}(\theta,\mathfrak{z}(\theta)) + \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-s)^{\alpha-1}[A\mathfrak{z}(s) + h(s,\mathfrak{z}(s)) + Bu(s)]ds. \quad (2.3)$$

Lemma 2.10. *Let $\mathfrak{z}$ satisfy (1.1) and (1.2). $\therefore$ for $\theta \in \mathcal{I}$, we have $\mathfrak{z}(\theta) \in \overline{\mathcal{D}_A}$. In particular, $\mathfrak{z}_0 + \phi(\mathfrak{z}) \in \overline{\mathcal{D}_A}$.*

Definition 2.11. For each $\theta \in \mathcal{Z}$ and $h \in \mathcal{Z}_0$, we define the integral solution of (1.1) and (1.2) as

$$\mathfrak{z}(\theta) = S_{\alpha,\beta}(\theta)[\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))] + \mathcal{P}(\theta,\mathfrak{z}(\theta)) + \lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda[A\mathcal{P}(s,\mathfrak{z}(s)) + Bu(s) + h(s,\mathfrak{z}(s))]ds, \quad (2.4)$$

where $\mathcal{B}_\lambda = \lambda(\lambda I - A)^{-1}$ such that $\mathcal{B}_\lambda\mathfrak{z} = \mathfrak{z}$ as $\lambda \to \infty$.

Lemma 2.12. *(see [16]). Let D be a closed and convex subset of $\mathcal{Z}$, $0 \in D$. If $F: \overline{D} \to \mathcal{Z}$ is continuous and of Mönch type, i.e. F satisfies the condition $\theta \subseteq \overline{D}$, θ is countable and $\theta \subseteq \overline{co}(\{0\} \cup F(\theta)) \Rightarrow \overline{\theta}$ is compact. Then F has at least one fixed point.*

3. Discussions on controllability

To ensure the outcomes, the subsequent hypotheses are introduced:

(H3) A function $h: (\mathcal{I}\times\mathcal{Z}) \to \mathcal{Z}$ satisfies:
(i). For all $\theta \in \mathcal{I}$, $h(\theta,\cdot): \mathcal{I} \to \mathcal{Z}$ is continuous, for $\mathfrak{z} \in \mathcal{Z}$, $h(\cdot,\mathfrak{z}): \mathcal{I} \to \mathcal{Z}$ is strongly measurable.
(ii). For functions $m_1 \in L^{\frac{1}{q}}(\mathcal{I}, R^+)$, $q \in (0,\alpha)$ and $\mathcal{L}_h: [0,\infty] \to (0,\infty)$, nondecreasing and continuous,

$$||h(\theta,\mathfrak{z}(\theta))|| \leq m_1(\theta)\mathcal{L}_h(\theta^{1-\vartheta}||\mathfrak{z}(\theta)||).$$

Also $\lim_{r\to\infty}\frac{\mathcal{L}_h(r)}{r} = \mathcal{L}_h^*$, for each $(\theta,\mathfrak{z}) \in \mathcal{I}\times\mathcal{Z}$ and $m_1^* = \max\{m_1(\theta)\}$.

(H4) There exists a constant $l_f{}^* > 0$, such that for any bounded $D_1 \subseteq \mathcal{Z}$, $\kappa(f(\theta,D_1)) \leq l_f{}^*\theta^{1-\vartheta}\kappa(D_1)$, almost everywhere $\theta \in \mathcal{I}$.

(H5) $\mathcal{P}: (\mathcal{I}\times\mathcal{Z}) \to \mathcal{Z}$ is bounded and Lipschitz continuous, which states that with some constants $m_g > 0$ and $\mathcal{L}_g \in (0,1)$ it satisfies

$$\|\mathcal{P}(\theta,\mathfrak{z}(\theta))\| \leq m_g \quad \text{and} \ \|\mathcal{P}(\theta,\mathfrak{z}_1(\theta)) - \mathcal{P}(\theta,\mathfrak{z}_2(\theta))\| \leq \mathcal{L}_g\|\mathfrak{z}_1 - \mathfrak{z}_2\|, \quad \text{for all } \theta \in \mathcal{I}.$$

(H6) There exists a constant $l_p{}^* > 0$, such that for any bounded $D_1 \subseteq \mathcal{Z}$, $\kappa(\mathcal{P}(\theta,D_1)) \leq l_p{}^*\theta^{1-\vartheta}\kappa(D_1)$, almost everywhere $\theta \in \mathcal{I}$.

(H7) For any constant $\mathcal{M}_3 > 0$ and for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{C}$,

$$||\phi(\mathfrak{z}_1) - \phi(\mathfrak{z}_2)|| \leq \mathcal{M}_3||\mathfrak{z}_1 - \mathfrak{z}_2||_{\mathcal{C}}.$$

(H8) $W: L^2(\mathcal{I},U) \to \mathcal{Z}$ defined as:

$$Wu = \lim_{\lambda\to+\infty}\int_0^a P_\alpha(a-s)\mathcal{B}_\lambda Bu(s)ds,$$

is invertible with the inverse operator denoted by W^{-1} which takes values in $L^2(\mathcal{I},U)|\ker W$ and for $\mathcal{M}_b, \mathcal{M}_w \geq 0$, provided that $\|B\| \leq \mathcal{M}_b$, $\|W^{-1}\| \leq \mathcal{M}_w$.

(H9) For some $l_u{}^* > 0$, such that $\kappa(u(z,\mu)) \leq l_u{}^*\theta^{1-\vartheta}\nu(z,\mu)\kappa(z(\mu))$, a.e $\mu \in \mathcal{I}$ with $\sup_{\theta\in\mathcal{I}}\int_0^\theta \nu(\theta,\mu)ds = \nu^* < \infty$.

Here, we model $u(\theta,\mathfrak{z})$ as:

$$u(\theta,\mathfrak{z}) = W^{-1}\Big[\mathfrak{z}_a - S_{\alpha,\beta}(a)[\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))] - \mathcal{P}(a,\mathfrak{z}(a)) - \lim_{\lambda\to\infty}\int_0^a P_\alpha(a-s)\mathcal{B}_\lambda[A\mathcal{P}(s,\mathfrak{z}(s)) + h(s,\mathfrak{z}(s))]ds\Big](\theta)$$

with

$$\begin{aligned}\|u(\theta,\mathfrak{z})\| &\leq ||W^{-1}\big[\mathfrak{z}_a - S_{\alpha,\beta}(a)[\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))] - \mathcal{P}(a,\mathfrak{z}(a)) - \lim_{\lambda\to\infty}\int_0^a P_\alpha(a-s)\mathcal{B}_\lambda[A\mathcal{P}(s,\mathfrak{z}(s)) + h(s,\mathfrak{z}(s))]ds\big](\theta)|| \\
&\leq \mathcal{M}_w\Big[\|\mathfrak{z}_a\| - \frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\|[\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))]\| - \|\mathcal{P}(a,\mathfrak{z}(a))\| - \frac{\mathcal{M}a^{\alpha-1}}{\Gamma(\alpha)}||\lim_{\lambda\to\infty}\int_0^a \mathcal{B}_\lambda[A\mathcal{P}(s,\mathfrak{z}(s)) + h(s,\mathfrak{z}(s))]ds||\Big] \\
&\leq \mathcal{M}_w\Big[\|\mathfrak{z}_a\| - \frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}} - m_g - \frac{\mathcal{M}a^{\alpha-1}\mathcal{M}_0}{\Gamma(\alpha)}\int_0^a\big[\|A\|m_g + m_1(s)L_h\big(s^{1-\vartheta}\|\mathfrak{z}(s)\|\big)\big]ds\Big] \\
&\leq \mathcal{M}_w\Big[\|\mathfrak{z}_a\| - \frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}} - m_g - \frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}\big[\|A\|m_g + m_1^*\mathcal{L}_h^*\big]\Big] \\
&\leq \mathcal{M}_w C_b^*,\end{aligned}$$

where $C_b^* = ||\mathfrak{z}_a|| - \frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}} - m_g - \frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g + m_1^*\mathcal{L}_h^*]$ and $\hat{\mathcal{M}} = ||\mathfrak{z}_0 + \phi(\mathfrak{z}) - \mathcal{P}(0,\mathfrak{z}(0))||$.

Let us consider the space $\mathcal{E} = \{\mathfrak{z} : \mathfrak{z} \in \mathcal{C}[\mathcal{I},\mathcal{Z}]\}$ equipped with the uniform convergence topology.

Theorem 3.1. *If (H1)–(H6) hold, then (1.1) and (1.2) has a unique solution provided that*

$$\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}} + m_g + \frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g + \mathcal{M}_b\mathcal{M}_wC_b^* + m_1^*\mathcal{L}_h^*] < \zeta^*, \quad (3.1)$$

and

$$\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+\frac{\mathcal{M}a^{\alpha}\mathcal{M}_0}{\Gamma(\alpha)}\Big[\|A\|\mathcal{L}_g+m_1(a)\mathcal{L}_h a^{1-\vartheta}+\Big[\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+\|A\|\mathcal{L}_g+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]\Big]<1. \quad (3.2)$$

Proof. Consider $\mathcal{B}_\iota(0,\mathcal{E})=\{\mathfrak{z}\in\mathcal{Z},||\mathfrak{z}||\le\zeta\}$. Then $\mathcal{B}_\iota(0,\mathcal{E})\subset\mathcal{C}[\mathcal{I},\mathcal{Z}]$ is a closed, bounded and convex set. For $\eta>0$, define the operator $\Gamma_\eta:\mathcal{B}_\iota(0,\mathcal{E})\to\mathcal{B}_\iota(0,\mathcal{E})$ as

$$\Gamma_\eta(\mathfrak{z}(\theta))=S_{\alpha,\beta}(\theta)[\mathfrak{z}_0+\phi(\mathfrak{z})-\mathcal{P}(0,\mathfrak{z}(0))]+\mathcal{P}(\theta,\mathfrak{z}(\theta))+\lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}(s))+Bu(s)+h(s,\mathfrak{z}(s))\Big]ds.$$

Here, we prove the existence and uniqueness by Banach contraction principle.

Step 1: Γ_η : maps $\mathcal{B}_\iota(0,\mathcal{E})$ into itself. For $\mathfrak{z}\in\mathcal{B}_\iota(0,\mathcal{E})$,

$$\begin{aligned}\|\Gamma_\eta(\mathfrak{z}(\theta))\|&\le||S_{\alpha,\beta}(\theta)[\mathfrak{z}_0+\phi(\mathfrak{z})-\mathcal{P}(0,\mathfrak{z}(0))]+\mathcal{P}(\theta,\mathfrak{z}(\theta))+\lim_{\lambda\to\infty}\\&\quad\times\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda[A\mathcal{P}(s,\mathfrak{z}(s))+Bu(s)+h(s,\mathfrak{z}(s))]ds||\\&\le\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}}+m_g\\&\quad+\frac{\mathcal{M}a^{\alpha-1}\mathcal{M}_0}{\Gamma(\alpha)}\int_0^a[\|A\|m_g]+\mathcal{M}_b\|u(s)\|+\|h(s,\mathfrak{z}(s))\|]ds\\&\le\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathcal{M}}+m_g\\&\quad+\frac{\mathcal{M}a^{\alpha}\mathcal{M}_0}{\Gamma(\alpha)}\big[\|A\|m_g+\mathcal{M}_b\mathcal{M}_wC_b^*+m_1^*\mathcal{L}_h^*\big]\\&\le\zeta^*.\end{aligned}$$

$\therefore$ Γ_η maps $\mathcal{B}_\iota(0,\mathcal{E})$ into itself.

Step 2: For some $\mathfrak{z},w\in\mathcal{B}_\iota(0,\mathcal{E})$,

$$\begin{aligned}||\Gamma_\eta(\mathfrak{z}(\theta))-\Gamma_\eta(w(\theta))||&\le||S_{\alpha,\beta}(\theta)(\phi(\mathfrak{z}))-\phi(w))||\\&\quad+||\mathcal{P}(\theta,\mathfrak{z}(\theta))-\mathcal{P}(\theta,w(\theta))||\\&\quad+\lim_{\lambda\to\infty}\Big|\Big|\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda\\&\quad\Big[A\mathcal{P}(s,\mathfrak{z}(s))+Bu(s,\mathfrak{z})+h(s,\mathfrak{z}(s))\Big]ds\\&\quad-\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda\Big[A\mathcal{P}(s,w(s))\\&\quad+Bu(s,w)+h(s,w(s))\Big]ds\Big|\Big|\end{aligned}$$

$$\begin{aligned}&\le\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3||\mathfrak{z}-w||+\mathcal{L}_g||\mathfrak{z}-w||+\frac{\mathcal{M}a^{\alpha-1}\mathcal{M}_0}{\Gamma(\alpha)}\\&\quad\times\Big[\int_0^a||A||\Big|\Big|\mathcal{P}(s,\mathfrak{z}(s))-\mathcal{P}(s,w(s))\Big|\Big|ds+\int_0^a\mathcal{M}_b\\&\quad\times\Big|\Big|u(s,\mathfrak{z}(s))-u(s,w(s))\Big|\Big|ds\\&\quad+\int_0^a\Big|\Big|h(s,\mathfrak{z}(s))-h(s,w(s))\Big|\Big|ds\Big]\\&\le\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3||\mathfrak{z}-w||+\mathcal{L}_g||\mathfrak{z}-w||\\&\quad+\frac{\mathcal{M}a^{\alpha}\mathcal{M}_0}{\Gamma(\alpha)}\Big[||A||\mathcal{L}_g+\mathcal{M}_b\mathcal{M}_w\\&\quad\times\Big[\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+||A||\mathcal{L}_g+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]\\&\quad+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]||\mathfrak{z}-w||\\&\le\Big[\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+\frac{\mathcal{M}a^{\alpha}\mathcal{M}_0}{\Gamma(\alpha)}\Big[||A||\mathcal{L}_g+\mathcal{M}_b\mathcal{M}_w\Big[\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3\\&\quad+\mathcal{L}_g+||A||\mathcal{L}_g+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]\Big]||\mathfrak{z}-w||\\&\le\mu^*||\mathfrak{z}-w||,\end{aligned}$$

where $\mu^*=\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+\frac{\mathcal{M}a^{\alpha}\mathcal{M}_0}{\Gamma(\alpha)}\Big[||A||\mathcal{L}_g+\mathcal{M}_b\mathcal{M}_w\Big[\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\mathcal{M}_3+\mathcal{L}_g+||A||\mathcal{L}_g+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]+m_1(a)\mathcal{L}_h a^{1-\vartheta}\Big]$.

Hence Γ_η is contraction. $\therefore$ Γ_η has a unique solution on $\mathcal{C}[\mathcal{I},\mathcal{Z}]$ by Banach contraction principle. □

Lemma 3.2. *If the hypotheses (H1)–(H9) hold, $\Gamma_\eta:\mathfrak{z}\in\mathcal{B}_\iota(0,\mathcal{E})$ is equicontinuous.*

Proof. By Lemma 2.8, $S_{\alpha,\beta}(\theta)$ is strongly continuous on $\mathcal{I}$.

For $\mathfrak{z}\in\mathcal{B}_\iota(0,\mathcal{E}),\theta_1,\theta_2\in\mathcal{I}$ and $\epsilon>0$ such that $0\le\epsilon<\theta_1<\theta_2\le a$ and there exists a $\delta>0$ such that if $0<|\theta_2-\theta_1|<\delta$, then

$$\begin{aligned}&\|\big(\Gamma_\eta\mathfrak{z}\big)(\theta_2)-\big(\Gamma_\eta\mathfrak{z}\big)(\theta_1)\|\\&\le||[\mathcal{P}(\theta_2,\mathfrak{z}(\theta_2))-\mathcal{P}(\theta_1,\mathfrak{z}(\theta_1))]+\lim_{\lambda\to\infty}\theta_2^{\vartheta-1}\\&\quad\int_0^{\theta_2}(\theta_2-s)^{\alpha-1}T_\alpha(\theta_2-s)\\&\quad\times B_\lambda A[\mathcal{P}(s,\mathfrak{z}(s))]ds-\theta_1^{\vartheta-1}\int_0^{\theta_1}(\theta_1-s)^{\alpha-1}T_\alpha(\theta_1-s)\\&\quad\times B_\lambda A[\mathcal{P}(s,\mathfrak{z}(s))]ds||+||\lim_{\lambda\to\infty}\theta_2^{\vartheta-1}\int_0^{\theta_2}(\theta_2-s)^{\alpha-1}T_\alpha(\theta_2-s)B_\lambda\\&\quad\times[h(s,\mathfrak{z}(s))]ds-\theta_1^{\vartheta-1}\int_0^{\theta_1}(\theta_1-s)^{\alpha-1}T_\alpha(\theta_1-s)B_\lambda[h(s,\mathfrak{z}(s)]ds||\\&\quad+||\lim_{\lambda\to\infty}\theta_2^{\vartheta-1}\int_0^{\theta_2}(\theta_2-s)^{\alpha-1}T_\alpha(\theta_2-s)B_\lambda[Bu(s)]ds\\&\quad-\theta_1^{\vartheta-1}\int_0^{\theta_1}(\theta_1-s)^{\alpha-1}T_\alpha(\theta_1-s)B_\lambda[Bu(s)]ds||\end{aligned}$$

$$\begin{aligned}&\le\mathcal{L}_g||\theta_2-\theta_1||+\Big|\Big|\lim_{\lambda\to\infty}\theta_2^{\vartheta-1}\int_{\theta_1}^{\theta_2}(\theta_2-s)^{\alpha-1}T_\alpha(\theta_2-s)\\&\quad\times B_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}(s))+h(s,\mathfrak{z}(s))+Bu(s)\Big]ds\Big|\Big|\\&\quad+\Big|\Big|\lim_{\lambda\to\infty}\int_0^{\theta_1}\Big[\theta_2^{\vartheta-1}(\theta_2-s)^{\alpha-1}-\theta_1^{\vartheta-1}(\theta_1-s)^{\alpha-1}\Big]T_\alpha(\theta_2-s)\\&\quad\times B_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}(s))+h(s,\mathfrak{z}(s))+Bu(s)\Big]ds\Big|\Big|\\&\quad+\Big|\Big|\lim_{\lambda\to\infty}\theta_1^{\vartheta-1}\int_0^{\theta_1-\epsilon}(\theta_1-s)^{\alpha-1}\Big[T_\alpha(\theta_2-s)-T_\alpha(\theta_1-s)\Big]\\&\quad\times B_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}(s))+h(s,\mathfrak{z}(s))+Bu(s)\Big]ds\Big|\Big|\\&\quad+\Big|\Big|\lim_{\lambda\to\infty}\theta_1^{\vartheta-1}\int_{\theta_1-\epsilon}^{\theta_1}(\theta_1-s)^{\alpha-1}\Big[T_\alpha(\theta_2-s)-T_\alpha(\theta_1-s)\Big]\\&\quad\times B_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}(s))+h(s,\mathfrak{z}(s))+Bu(s)\Big]ds\Big|\Big|.\end{aligned}$$

Using absolute continuity by virtue of the Lebesgue convergence theorem and for ϵ sufficiently small $||(\Gamma_\eta\mathfrak{z})(\theta_2)-(\Gamma_\eta\mathfrak{z})(\theta_1)||\to 0$ *as* $\theta_2\to\theta_1$. Hence Γ_η is equicontinuous. □

Lemma 3.3. *If the hypotheses (H1)–(H9) hold, $\Gamma_\eta:y\in\mathcal{B}_\iota(0,\mathcal{E})$ is continuous provided that*

$$k_b^*\left[\left[\frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]+m_g\right]\left[1+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\right]\right]<1. \tag{3.3}$$

Proof. Step 1: $\Gamma_\eta(\mathcal{B}_\iota(0,\mathcal{E}))\subset\mathcal{B}_\iota(0,\mathcal{E})$.

Suppose if it fails, for all $\iota>0$, and $\mathfrak{z}^\iota\in\mathcal{B}_\iota(0,\mathcal{E})$, $\theta^\iota\in\mathcal{I}$ yields $\iota<||(\Gamma_\eta\mathfrak{z}^\iota)(\theta^\iota)||_{\mathcal{C}}$.

Consider $0<\theta<\theta^\iota$ such that $\lim_{\iota\to\infty}\frac{\iota^*}{\iota}=k_b^*$, $||\mathfrak{z}^\iota||_{\mathcal{Z}}\le\iota^*$, we get

$$\begin{aligned}
\iota &< ||(\Gamma_\eta\mathfrak{z}^\iota)(\theta^\iota)||_{\mathcal{C}}\\
&< \sup_{0\le\theta^\iota\le a}\Bigg[||S_{\alpha,\beta}(\theta^\iota)[\mathfrak{z}_0+\phi(\mathfrak{z}^\iota)-\mathcal{P}(0,\mathfrak{z}(0))]||_{\mathcal{Z}}\\
&\quad+||\mathcal{P}(\theta^\iota,\mathfrak{z}^\iota(\theta^\iota))||_{\mathcal{Z}}\\
&\quad+\left|\left|\lim_{\lambda\to\infty}\int_0^{\theta^\iota}AP_\alpha(\theta^\iota-s)\mathcal{B}_\lambda\mathcal{P}(s,\mathfrak{z}^\iota(s))ds\right|\right|_{\mathcal{Z}}\\
&\quad+\left|\left|\lim_{\lambda\to\infty}\int_0^{\theta^\iota}P_\alpha(\theta^\iota-s)\mathcal{B}_\lambda Bu(s,\theta^\iota)ds\right|\right|_{\mathcal{Z}}\\
&\quad+\left|\left|\lim_{\lambda\to\infty}\int_0^{\theta^\iota}P_\alpha(\theta^\iota-s)\mathcal{B}_\lambda h(s,\mathfrak{z}^\iota(s))ds\right|\right|_{\mathcal{Z}}\Bigg]\\
&< \frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\Big[||\mathfrak{z}_0||+||\phi(\mathfrak{z}^\iota)||-m_g\Big]_{\mathcal{Z}}+m_g\iota^*\\
&\quad+\frac{||A||\mathcal{M}a^\alpha\mathcal{M}_0m_g\iota^*}{\Gamma(\alpha)}+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\iota^*m_1^*\mathcal{L}_h^*}{\Gamma(\alpha)}\\
&\quad+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\Bigg[||\mathfrak{z}_a||+\frac{\mathcal{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\\
&\quad\Big[||\mathfrak{z}_0||+||\phi(\mathfrak{z}^\iota)||-m_g\Big]+m_g\iota^*\\
&\quad+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\iota^*}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]\\
&< \frac{\mathcal{M}a^{\vartheta-1}\hat{\mathcal{M}}}{\Gamma(\vartheta)}+m_g\iota^*+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\iota^*}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]\\
&\quad+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\\
&\quad\Bigg[||\mathfrak{z}_a||+\hat{\mathcal{M}}\iota^*+m_g\iota^*+\frac{\mathcal{M}a^{\alpha-1}\mathcal{M}_0\iota^*}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]\Bigg].
\end{aligned}$$

Dividing by ι and taking $\iota\to\infty$,

$$\begin{aligned}
1 &< k_b^*\Bigg[\frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]+m_g+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\\
&\quad[m_g+\frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]\Bigg]\\
&< k_b^*\Bigg[\frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]\left[1+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\right]\\
&\quad+m_g\left[1+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\right]\Bigg]\\
&< k_b^*\Bigg[\left[\frac{\mathcal{M}a^\alpha\mathcal{M}_0}{\Gamma(\alpha)}[||A||m_g+m_1^*\mathcal{L}_h^*]+m_g\right]\\
&\quad\left[1+\frac{\mathcal{M}a^\alpha\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\right]\Bigg],
\end{aligned}$$

which contradicts the assumption (3.3). Hence for $\iota>0$, $\Gamma_\eta(\mathcal{B}_\iota(0,\mathcal{E}))\subset\mathcal{B}_\iota(0,\mathcal{E})$.

Step 2: Γ_η is continuous on $\mathcal{B}_\iota(0,\mathcal{E})$.

Let $\mathfrak{z}_k,\mathfrak{z}$ be in $\mathcal{B}_\iota(0,\mathcal{E})$, for each $k=1,2,\dots$ provided $\lim_{k\to\infty}||\mathfrak{z}_k-\mathfrak{z}||\to0$ and $\lim_{k\to\infty}\mathfrak{z}_k(\theta)\to\mathfrak{z}(\theta)$, for all $\theta\in\mathcal{I}$. $\therefore$

$$\lim_{k\to\infty}||h(\theta,\mathfrak{z}_k(\theta))-h(\theta,\mathfrak{z}(\theta))||\to0.$$

Using $(H1)$, for $\theta\in\mathcal{I}$,

$$(\theta-s)^{\alpha-1}||h(\theta,\mathfrak{z}_k(\theta))-h(\theta,\mathfrak{z}(\theta))||\le(\theta-s)^{\alpha-1}m_1(s)\mathcal{L}_h\Big[||\mathfrak{z}_k-\mathfrak{z}||\Big]\ \text{a.e}\ s\in(0,\theta).$$

Also for $s\in(0,\theta)$ and $\theta\in[0,a]$, $(\theta-s)^{\alpha-1}m_1(s)\mathcal{L}_h\Big[||\mathfrak{z}_k-\mathfrak{z}||\Big]$ is integrable. Moreover,

$$\int_0^\theta(\theta-s)^{\alpha-1}||h(s,\mathfrak{z}_k(s))-h(s,\mathfrak{z}(s))||ds\to0\ \text{as}\ k\to\infty. \tag{3.4}$$

Hence

$$\begin{aligned}
||(\Gamma_\eta\mathfrak{z}_k)(\theta)-(\Gamma_\eta\mathfrak{z})(\theta)|| &\le a^{\vartheta-1}\Bigg|\Bigg|\lim_{\lambda\to\infty}\int_0^\theta(\theta-s)^{\alpha-1}B_\lambda T_\alpha(\theta-s)\\
&\quad\Big[A[\mathcal{P}(s,\mathfrak{z}_k(s))-\mathcal{P}(s,\mathfrak{z}(s))]\\
&\quad+[h(s,\mathfrak{z}_k(s))-h(s,\mathfrak{z}(s))]\\
&\quad+B[u_{\mathfrak{z}_k}-u_{\mathfrak{z}}]\Big]ds\Bigg|\Bigg|\\
&\le\frac{\mathcal{M}a^{\vartheta-1}\mathcal{M}_0}{\Gamma(\alpha)}\Bigg|\Bigg|\int_0^\theta(\theta-s)^{\alpha-1}\\
&\quad\Big[A[\mathcal{P}(s,\mathfrak{z}_k(s))-\mathcal{P}(s,\mathfrak{z}_k(s))]\\
&\quad+[h(s,\mathfrak{z}_k(s))-h(s,\mathfrak{z}(s))]\\
&\quad+B[u_{\mathfrak{z}_k}-u_{\mathfrak{z}}]\Big]ds\Bigg|\Bigg|.
\end{aligned}\tag{3.5}$$

By (3.4) and (3.5),

$$||(\Gamma_\eta\mathfrak{z}_k)(\theta)-(\Gamma_\eta\mathfrak{z})(\theta)||\to0\ \text{as}\ n\to\infty.$$

Hence Γ_η is continuous on $\mathcal{B}_\iota(0,\mathcal{E})$.
□

Theorem 3.4. *If the hypotheses (H1)–(H9) hold, then the system (1.1) and (1.2) is controllable on $\mathcal{I}$ provided that*

$$\begin{aligned}
&l_p{}^*\theta^{1-\vartheta}\kappa(\mathbf{S})+\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}(l_p{}^*+l_f{}^*)\theta^{1-\vartheta}\left[1+\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}l_u{}^*\nu^*\right]\\
&\int_0^a(a-s)^{\alpha-1}\kappa(\mathbf{S})ds<r.
\end{aligned}\tag{3.6}$$

Proof. In order to satisfy Mönch condition, construct the countable subset $\mathbf{S}$ of $\mathcal{B}_\iota(0,\mathcal{E})$ and $\mathbf{S}\subset\overline{\text{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))$, then we prove $\kappa(\mathbf{S})=0$.

Let $\mathbf{S}=\{\mathfrak{z}_n\}_{n=1}^\infty$. By Lemma 3.2, we note that $\Gamma_\eta\{\mathfrak{z}_n\}_{n=1}^\infty$ is equicontinuous on $\mathcal{I}$, then

$\mathbf{S}\subset\overline{\text{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))$ is also equicontinuous on $\mathcal{I}$.

$$\begin{aligned}
&\kappa\Big(u\big(\theta,\{\mathfrak{z}_n\}_{n=1}^\infty\big)\Big)\\
&\le\kappa\Bigg\{W^{-1}\Bigg[\mathfrak{z}_a-S_{\alpha,\beta}(a)[\mathfrak{z}_0+\phi(\mathfrak{z})-\mathcal{P}(0,\mathfrak{z}(0))]-\mathcal{P}(a,\mathfrak{z}(a))-\lim_{\lambda\to\infty}\\
&\quad\times\int_0^a(a-s)^{\alpha-1}T_\alpha(a-s)\mathcal{B}_\lambda\Big[A\mathcal{P}\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)+h\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\Big]ds\Bigg]\Bigg\}\\
&\le l_u{}^*\nu(\theta)\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_w}{\Gamma(\alpha)}\\
&\quad\times\int_0^a(a-s)^{\alpha-1}\kappa\Big(\Big[A\mathcal{P}\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)+h\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\Big]\Big)ds
\end{aligned}$$

$$\leq l_u{}^*\nu(\theta)\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_w}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\big(l_p{}^*+l_f{}^*\big)s^{1-\vartheta}\kappa\left(\{\mathfrak{z}_n(s)\}_{n=1}^\infty\right)ds.$$

Also by Lemma 2.7,

$$\begin{aligned}
&\kappa\left(\Gamma_\eta\left(\{\mathfrak{z}_n(\theta)\}_{n=1}^\infty\right)\right)\\
&=\kappa\left\{S_{\alpha,\beta}(\theta)[\mathfrak{z}_0+\phi(\mathfrak{z})-\mathcal{P}(0,\mathfrak{z}(0))]\right.\\
&+\mathcal{P}\big(\theta,\{\mathfrak{z}_n(\theta)\}_{n=1}^\infty\big)+\lim_{\lambda\to\infty}\int_0^\theta(\theta-s)^{\alpha-1}T_\alpha(\theta-s)\mathcal{B}_\lambda\\
&\left.\big[A\mathcal{P}\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)+Bu(s)+h\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\big]ds\right\}\\
&\leq\kappa\left(\mathcal{P}\big(\theta,\{\mathfrak{z}_n(\theta)\}_{n=1}^\infty\big)+\lim_{\lambda\to\infty}\int_0^\theta(\theta-s)^{\alpha-1}T_\alpha(\theta-s)\right.\\
&\times\mathcal{B}_\lambda\big[A\mathcal{P}\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)+h\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\big]ds\\
&\left.+\lim_{\lambda\to\infty}\int_0^\theta(\theta-s)^{\alpha-1}T_\alpha(\theta-s)\mathcal{B}_\lambda\big[Bu\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\big]ds\right)\\
&\leq l_p{}^*\theta^{1-\vartheta}\kappa\left(\{\mathfrak{z}_n(\theta)\}_{n=1}^\infty\right)+\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\big(l_p{}^*+l_f{}^*\big)s^{1-\vartheta}\\
&\times\kappa\left(\{\mathfrak{z}_n(s)\}_{n=1}^\infty\right)ds+\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_b}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\kappa\\
&\big(u\big(s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty\big)\big)ds\\
&\times\kappa\left(\{\mathfrak{z}_n(s)\}_{n=1}^\infty\right)ds+\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\\
&\times\left[l_u{}^*\nu(s)\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}\int_0^a(a-\xi)^{\alpha-1}\big(l_p{}^*+l_f{}^*\big)s^{1-\vartheta}\right.\\
&\left.\kappa\left(\{\mathfrak{z}_n(\xi)\}_{n=1}^\infty\right)d\xi\right]ds\\
&\times\int_0^a(a-s)^{\alpha-1}\left[l_u{}^*\nu(s)\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}\int_0^a\right.\\
&\left.(a-\xi)^{\alpha-1}\big(l_p{}^*+l_f{}^*\big)s^{1-\vartheta}\kappa\left(\{\mathfrak{z}_n(\xi)\}_{n=1}^\infty\right)d\xi\right]ds\\
&\leq l_p{}^*\theta^{1-\vartheta}\kappa(\mathbf{S})+\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}\big(l_p{}^*+l_f{}^*\big)\theta^{1-\vartheta}\\
&\times\left[1+\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}l_u{}^*\nu^*\right]\int_0^a(a-s)^{\alpha-1}\kappa(\mathbf{S})ds.
\end{aligned}$$

Using Mönch condition,

$$\begin{aligned}
\kappa(\mathbf{S})&\leq\overline{\text{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))=\kappa(\Gamma_\eta(\mathbf{S}))\\
&\leq l_p{}^*\theta^{1-\vartheta}\kappa(\mathbf{S})+\frac{2\mathcal{M}\mathcal{M}_0}{\Gamma(\alpha)}(l_p{}^*+l_f{}^*)\theta^{1-\vartheta}\\
&\left[1+\frac{2\mathcal{M}\mathcal{M}_0\mathcal{M}_b\mathcal{M}_w}{\Gamma(\alpha)}l_u{}^*\nu^*\right]\int_0^a(a-s)^{\alpha-1}\kappa(\mathbf{S})ds.
\end{aligned}$$

Using Grownwall's inequality, we conclude that $\kappa(\mathbf{S})=0$. By Lemma 2.12, we observe that Γ_η has a fixed point in $\mathcal{B}_\iota(0,\mathcal{E})$. Hence the system (1.1) and (1.2) has a fixed point satisfying $\mathfrak{z}(a)=\mathfrak{z}_a$. Therefore, (1.1) and (1.2) is controllable on $[0,a]$. □

4. Results on optimal control

Consider the Lagrange problem (LP): Find a control $(\mathfrak{z}^0,u^0)\in C_{1-\vartheta}([0,b],X)\times U_{ad}$ provided that $\mathcal{J}(\mathfrak{z}^0,u^0)\leq\mathcal{J}(\mathfrak{z},u)$, for all $u\in U_{ad}$ with

$$\mathcal{J}(\mathfrak{z},u)=\int_0^b\mathcal{L}(\theta,\mathfrak{z}(\theta),u(\theta))d\theta$$

where U_{ad} denotes an admissible control set. Here $\mathfrak{z}$ is the solution of the system (1.1) and (1.2) corresponding to the control $u\in U_{ad}$. To analyze the problem (LP) we assume the subsequent hypotheses:

(H10) (i) The functional $\mathcal{L}:J\times X\times U\to\mathbb{R}\cup\{\infty\}$ is Borel measurable;
(ii) $\mathcal{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $X\times U$ for almost all $t\in J$;
(iii) $\mathcal{L}(t,x,\cdot)$ is convex on U for each $x,y\in X$ for almost all $t\in J$;
(iv) There exist constants $d\geq0$, $j>0$, μ is nonnegative and $\mu\in L^P(J,\mathbb{R})$ such that

$$\mathcal{L}(\theta,\mathfrak{z},u)\geq\mu(\theta)+d||\mathfrak{z}||_C+j||u||_U^p.$$

Theorem 4.1. *If (H1)–(H10) hold, then LP has at least one optimal pair.*

Proof. If $\inf\{\mathcal{J}(\mathfrak{z},u)|(\mathfrak{z},u)\in C_{1-\vartheta}(J,X)\times U_{ad}\}=+\infty$, then the proof is trivial. Suppose

$$\inf\{\mathcal{J}(\mathfrak{z},u)|(\mathfrak{z},u)\in C_{1-\vartheta}(J,X)\times U_{ad}\}=\gamma<+\infty.$$

By (H6), we get $\gamma>-\infty$. By infimum definition, there exists a minimizing sequential pair $\{(\mathfrak{z}^n,u^n)\}\subset A_{ad}$, the set of all admissible state control pairs $(\mathfrak{z},u)$ such that $\mathcal{J}(\mathfrak{z}^n,u^n)\to\gamma$ as $n\to+\infty$. Since $\{u^n\}\subseteq U_{ad}$ for all $n\in N$, it is clear that $\{u^n\}$ is bounded on $L^P(J,U)$. Using the reflexive property, we show that there exists a sub-sequence, $\{u^0\}\in L^P(J,U)$ such that $\{u^n\}$ weakly converges to $\{u^0\}$ in $L^P(J,U)$. Since U_{ad} is closed and convex, by Mazur's lemma, $u^0\in U_{ad}$.

Let $\{z^n\}$ be the solution sequence of the integral equation

$$\mathfrak{z}^n(\theta)=S_{\alpha,\beta}(\theta)[\mathfrak{z}_0+\phi(\mathfrak{z}^n)-\mathcal{P}(0,\mathfrak{z}(0))]+\mathcal{P}(\theta,\mathfrak{z}^n(\theta))+\lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda\Big[A\mathcal{P}(s,\mathfrak{z}^n(s))+Bu^n(s)+h(s,\mathfrak{z}^n(s))\Big]ds.$$

By using Lemma 2.8, the boundedness of $\{u^n\}$ and following Theorem 3.4, $\{\mathfrak{z}^n\}$ is a relatively compact subset of $C_{1-\vartheta}(J,X)$. Therefore there is a function $\mathfrak{z}^0\in C_{1-\vartheta}(J,X)$ such that

$$\mathfrak{z}^n\to\mathfrak{z}^0\in C_{1-\vartheta}(J,X). \tag{4.1}$$

Using (H3), (H5), (H7) and Eq. (4.1) with dominated convergence theorem, we get

$$\int_0^\theta P_\alpha(\theta-s)h(s,\mathfrak{z}^n(s))ds\to\int_0^\theta P_\alpha(\theta-s)h(s,\mathfrak{z}^0(s))ds,$$

$$\int_0^\theta P_\alpha(\theta-s)A\mathcal{P}(s,\mathfrak{z}^n(s))ds\to\int_0^\theta P_\alpha(\theta-s)A\mathcal{P}(s,\mathfrak{z}^0(s))ds,$$

$$\text{and}\quad\phi(\mathfrak{z}^n)\to\phi(\mathfrak{z}^0).$$

By above-said terms, we infer that

$$\begin{aligned}
\mathfrak{z}^n(\theta)\to\mathfrak{z}^0(\theta)&=S_{\alpha,\beta}(\theta)[\mathfrak{z}_0+\phi(\mathfrak{z}^0)-\mathcal{P}(0,\mathfrak{z}(0))]+\mathcal{P}(\theta,\mathfrak{z}^0(\theta))\\
&+\lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-s)\mathcal{B}_\lambda\\
&\Big[A\mathcal{P}(s,\mathfrak{z}^0(s))+Bu^0(s)+h(s,\mathfrak{z}^0(s))\Big]ds,
\end{aligned}$$

where $\mathfrak{z}^0$ represents the solution sequence of the system (1.1) and (1.2) corresponding to u^0. Using (H10) and Balder's theorem, we get

$$(\mathfrak{z},u)\to\int_0^b\mathcal{L}(\theta,\mathfrak{z}(\theta),u(\theta))d\theta$$

is sequentially lower semicontinuous in the weak topology of $L^P(J,X)$. We conclude that $\mathcal{J}$ is weakly lower semicontinuous on $L^P(J,X)$. By (H10(iv)), $\mathcal{J}$ attains its infimum at $u^0\in U_{ad}$, that is,

$$\lim_{n\to\infty}\int_0^b\mathcal{L}(\theta,\mathfrak{z}^n(\theta),u^n(\theta))d\theta\geq\int_0^b\mathcal{L}(\theta,\mathfrak{z}^0(\theta),u^0(\theta))d\theta\geq\gamma.$$

□

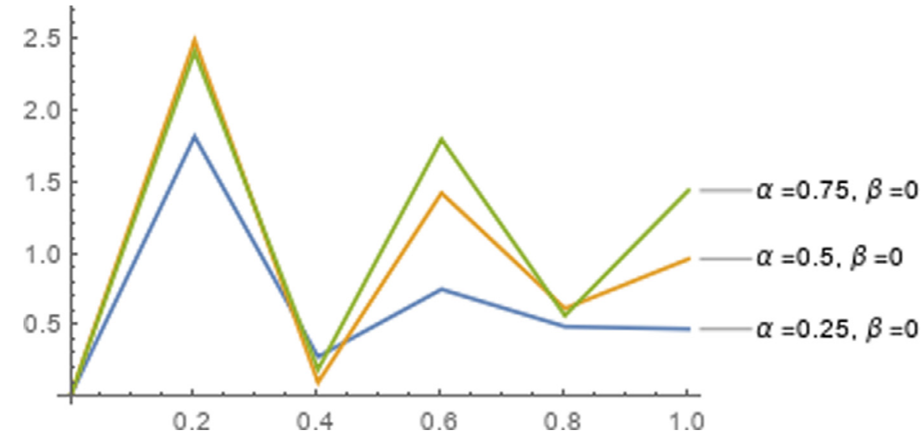

Fig. 1. Numerical approximation for R-L (Hilfer with $\beta = 0$) form.

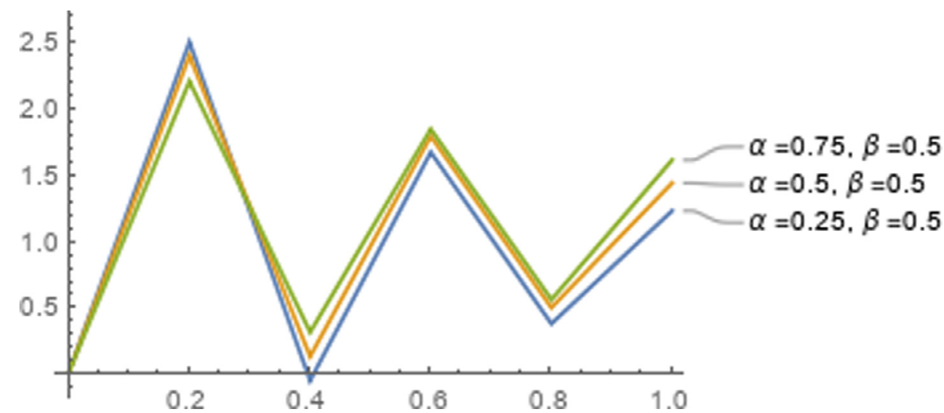

Fig. 2. Numerical approximation for Hilfer ($\beta = 0.5$) form.

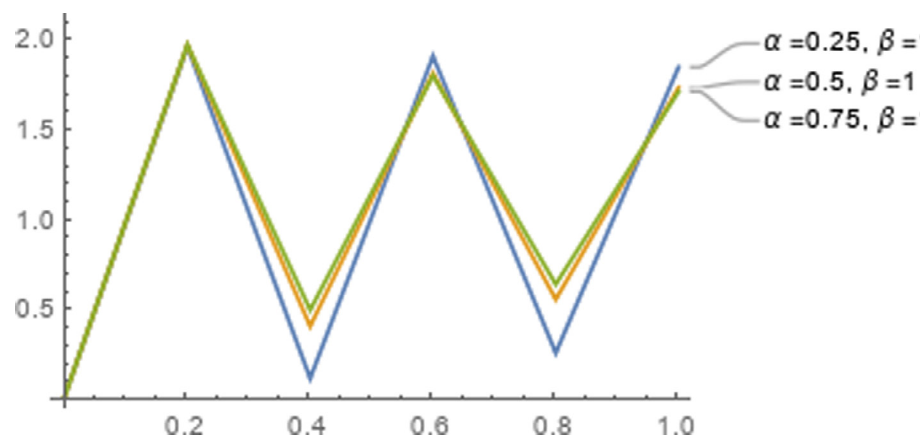

Fig. 3. Numerical approximation for Caputo (Hilfer with $\beta = 1$) form.

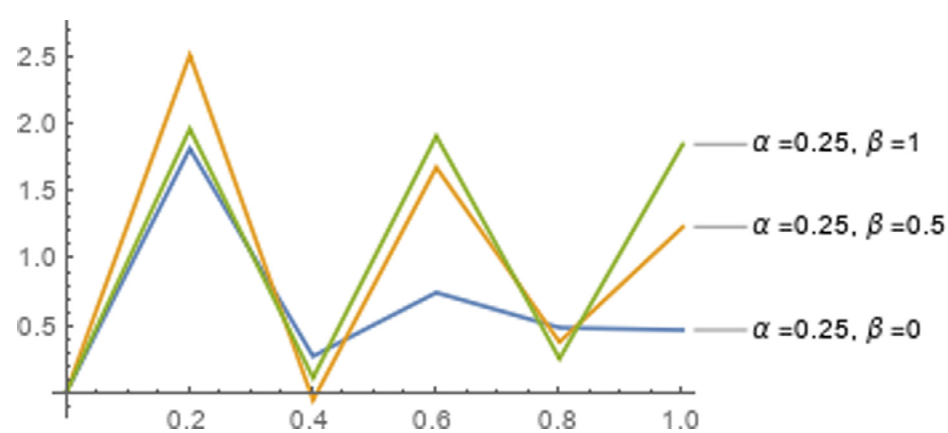

Fig. 4. Numerical approximation for $\alpha = 0.25$.

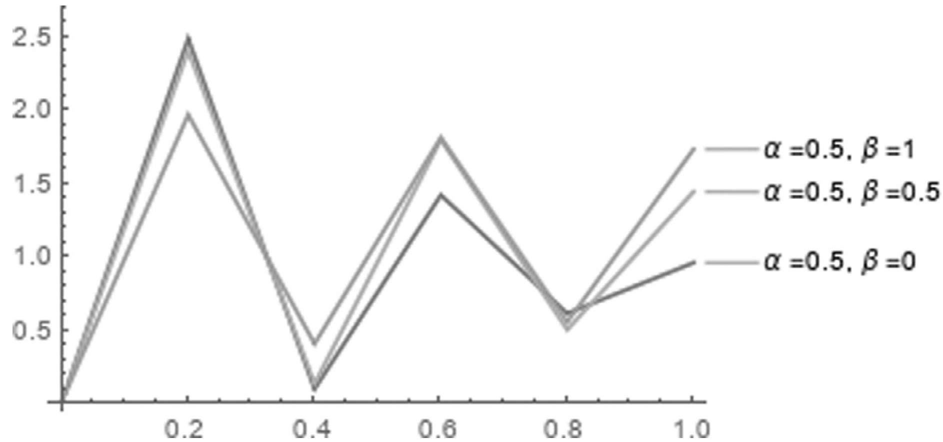

Fig. 5. Numerical approximation for $\alpha = 0.5$.

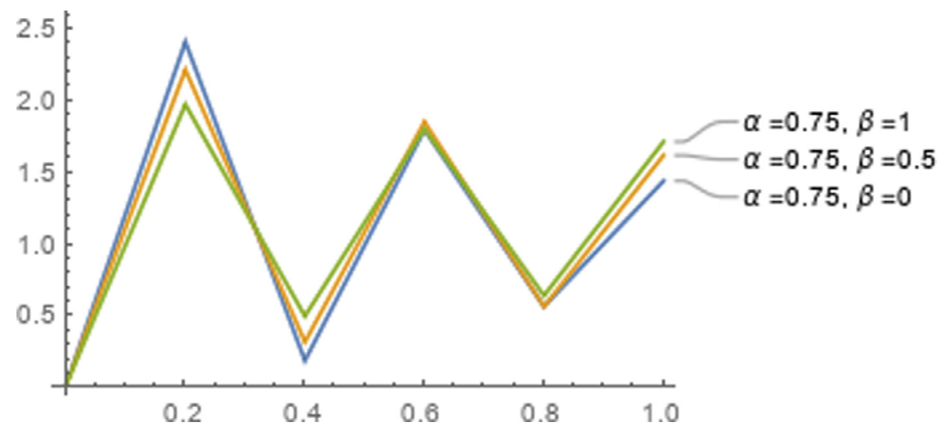

Fig. 6. Numerical approximation for $\alpha = 0.75$.

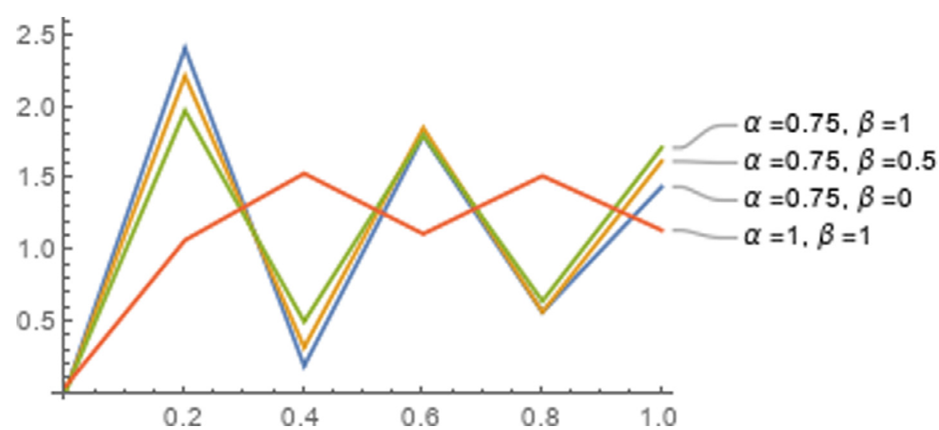

Fig. 7. Numerical approximation for $\alpha = 0.75, \alpha = 1$.

5. Numerical analysis

Consider the problem

$$\mathcal{D}_{0+}^{\alpha,\beta}[y(t) - \frac{\sin(y(t))}{40}] = Ay(t) + \frac{e^{-t}\sin(y(t))}{4}, \tag{5.1}$$

$$I_{0+}^{(1-\alpha)(1-\beta)}y(0) = 1 + \cos y, \qquad t \in \mathcal{I} = [0,1], \tag{5.2}$$

Consider $D(A) = \{y \in \mathcal{C}^2([0,1], R) : y(0) = y(1) = 0\}$, $Ay = y''$ where $\mathcal{H} : \mathcal{C}([0,1], R)$ provided with the uniform topology and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$.

A successive approximation of (5.1) and (5.2) is

$$y_n(t) = \frac{\left[1 + \cos y + \frac{\sin(y(0))}{40}\right]t^{\theta-1}}{\Gamma(\theta)} + \frac{\sin(y_{n-1}(t))}{40} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{e^{-s}\sin(y_{n-1}(s))}{4}ds,$$

where $\vartheta = \alpha + \beta - \alpha\beta$ and n varies from 1 to 6.

By Remark 2.4, we analyze the numerical approximation for existence of three types of solutions. Figs. 1–10 represent the solutions in Riemann–Liouville, Hilfer and Caputo's forms and

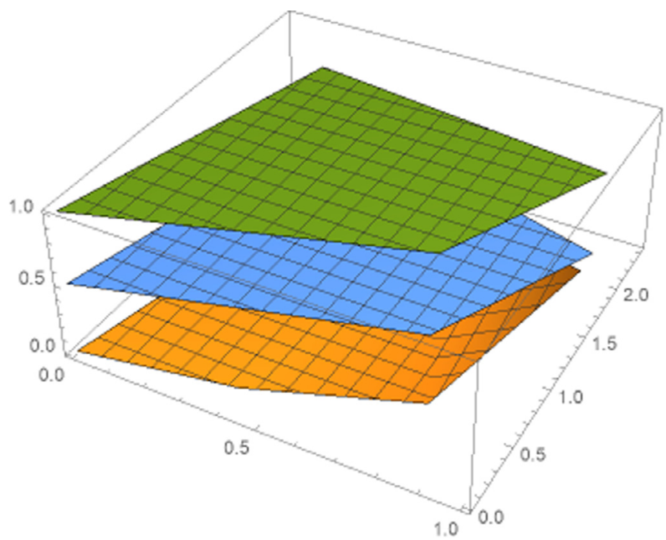

Fig. 8. Numerical approximation for $\alpha = 0.75$.

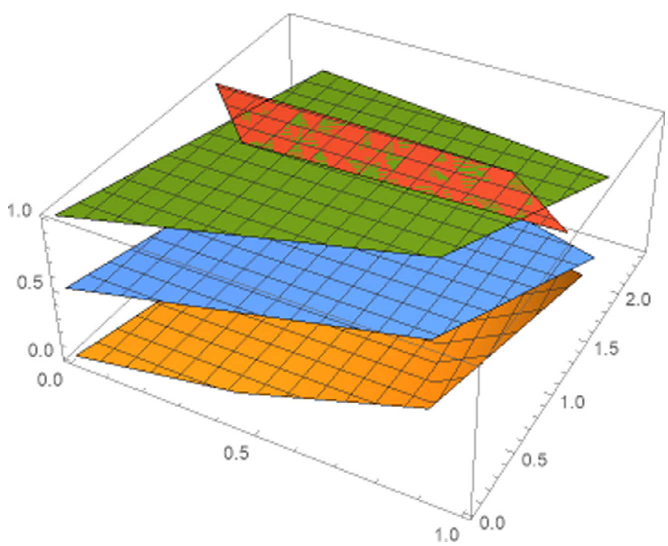

Fig. 9. Numerical approximation for $\alpha = 0.75, \alpha = 1$.

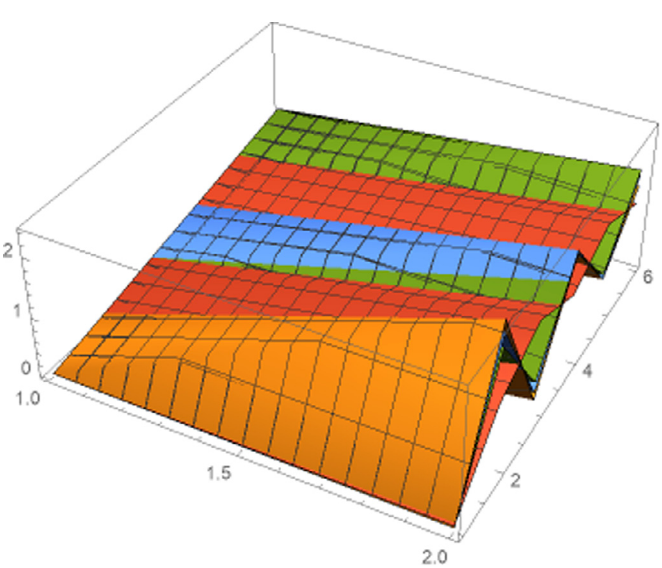

Fig. 10. Numerical approximation for $\alpha = 0.75$.

also the comparison among them for the different values of $\alpha = 0.25, 0.5, 0.75$ and $\beta = 0, 0.5, 1$.

6. Conclusion

This manuscript illustrates the controllability of neutral Hilfer fractional derivative with non-dense domain in a Banach space using semigroup theory, fixed point approach, and fundamentals of fractional calculus. Our theorem ensures the effectiveness of controllability and optimal control of the system concerned. Finally, numerical analyses have been done to compare the solution in different criteria of parameters. One can elevate this theory via some additional inclusions and different fixed point approaches.

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CRediT authorship contribution statement

Kottakkaran Sooppy Nisar: Conceptualization, Writing - original draft, Software, Formal analysis, Writing - review & editing. **K. Jothimani:** Writing - original draft, Software, Validation. **K. Kaliraj:** Formal analysis, Writing - original draft, Software. **C. Ravichandran:** Conceptualization, Writing - original draft, Methodology, Software, Supervision, Writing - review & editing.

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