MORE ON NANO PRE-NEIGHBOURHOODS IN NANO TOPOLOGICAL SPACES

P. Sathishmohan[1], V. Rajendran[2], P.K. Dhanasekaran[3] and C. Vignesh Kumar[4]

Department of Mathematics,
Kongunadu Arts and Science College, Coimbatore−641 029.
Email :[1] iiscsathish@yahoo.co.in, [3]dhanasekaranmath@gmail.com

Abstract

The basic objective of this paper is to introduce and investigate the properties of nano pre-neighbourhoods, nano pre-interior, nano pre-limit point, nano pre-derived set, nano pre-frontier, nano pre-regular in nano topological spaces and obtain some of its basic results.

1. Introduction

The notion of nano topology was introduced by Lellis Thivagar[1] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it and also defined nano-closed sets, nano-interior and nano-closure of a set. He also introduced weak form of nano-open sets namely nano α-open sets, *NS*-open sets and *NP*-open sets. In this paper we defined nano pre-neighbourhood, nano pre-interior, nano pre-limit point, nano pre-derived set, nano pre-frontier, nano pre-regular and obtained some of its basic results.

2. Preliminaries

Definition 2.1. [1] *Let U be the universe, R be an equivalence relation on U and* $\tau_R(X) = \{U,\phi,L_R(X),U_R(X),B_R(X)\}$*, where* $X \subseteq U$*. Then* $\tau_R(X)$ *satisfies the following axioms:*

- *U and* $\phi \in \tau_R(X)$.
- *The union of the elements of any subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.
- *The intersection of the elements of any finite subcollection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

Then $\tau_R(X)$ *is a topology on U called the nano topology on U with respect to X. We call* $(U,\tau_R(X))$ *is a nano topological space. The elements of* $\tau_R(X)$ *are called as nano-open sets. The complement of the nano-open sets are called nano-closed sets.*

Definition 2.2. [1] *Let* $(U,\tau_R(X))$ *be a nano topological space and* $A \subseteq U$*. Then A is said to be*

- *nano semi-open if* $A \subseteq Ncl(Nint(A))$.
- *nano pre-open if* $A \subseteq Nint(Ncl(A))$.
- *nano α-open if* $A \subseteq Nint(Ncl(Nint(A)))$.
- *nano semi pre-open if* $A \subseteq Ncl(Nint(Ncl(A)))$.
- *Nr-open if* $A = Nint(Ncl(A))$.

$NSO(U,X)$, $NPO(U,X)$, $\tau^{\alpha}_{R}(X)$, $NSPO(U,X)$ and $NRO(U,X)$ respectively denote the families of all nano semi-open, nano pre-open, nano α-open, nano semi pre-open and nano regular-open subsets of U. Let $(U,\tau_R(X))$ be a nano topological space and $A \subseteq U$, A is said to be nano semi-closed, nano pre-closed, nano α-closed, nano semi pre-closed and nano regular-closed if its complement is respectively nano semi-open, nano pre-open, nano α-open, nano semi pre-open and nano regular-open.

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Remark 2.3. [2] *If τ_R is the nano topology on U with respect to X, then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for τ_R.*

Definition 2.4. [1] *If (U,τ_R) is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then*

(1) *The nano-interior of A is defined as the union of all nano-open subsets of A and is denoted by $Nint(A)$. That is, $Nint(A)$ is the largest nano-open subset of A.*
(2) *The nano-closure of A is defined as the intersection of all nano-closed sets containing A and is denoted by $Ncl(A)$. That is, $Ncl(A)$ is the smallest nano-closed set containing A.*

Definition 2.5. [1] *Let $(U,\tau_R(X))$ be a nano topological space and $A \subseteq U$. Then A is said to be nano pre-open if $A \subseteq Nint(Ncl(A))$. It is denoted by $NPO(U)$. The complement of nano pre-open set is called nano pre-closed and it is denoted by $NPF(U)$.*

3. Nano Pre-Neighbourhoods

Definition 3.1. *A subset $M_x \subset U$ is called a nano pre-neighbourhood of a point $x \in U$ iff there exists a $A \in NPO(U)$ such that $x \in A \subset M_x$ and a point x is called nano pre-neighbourhood point of the set A.*

Definition 3.2. *The family of all nano pre-neighbourhoods of the point $x \in U$ is called nano pre-neighbourhood of U.*

Example 3.3. *Let $U = \{a,b,c,d\}$, $U/R = \{\{a,c\},\{b\},\{d\}\}$, $X = \{b,c\}$ and $\tau_R(X) = \{U,\phi,\{b\},\{a,b,c\},\{a,c\}\}$. $NPO(U) = \{U,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$.*
Then $NP - nbd(a) = \{U,\phi,\{a\},\{a,b\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$
$NP - nbd(b) = \{U,\phi,\{b\},\{a,b\},\{b,d\},\{a,b,c\},\{a,b,d\}\}$
$NP - nbd(c) = \{U,\phi,\{a,b,c\},\{a,c,d\}\}$
$NP - nbd(d) = \{U,\phi,\{d\},\{a,d\},\{b,d\},\{a,b,d\},\{a,c,d\}\}$

Lemma 3.4. *Let $\{B_i \mid i \in I\}$ be a collection of nano pre-open sets in a nano topological space U, then $\cup B_i \in NPO(U)$.*

Definition 3.5. *The union of all nano pre-open sets which are contained in A is called the nano pre-interior of A and is denoted by $Npint(A)$ or by NA_*. As the union of nano pre-open sets is nano pre-open, NA_* is nano pre-open.*

Definition 3.6. *The intersection of nano pre-closed sets containing a set A is called the nano pre-closure of A and is denoted by Npcl(A) or by* NA^*.

Lemma 3.7. *Let A and B be subsets of a space U. Then the following hold for the nano pre-closure operator.*

(1) $A = NA^*$.

(2) $NA^* \subset NB^*$ *if* $A \subset B$.

(3) $(A^*)^* = A^*$.

(4) A^* *is pre-closed set in X.*

Lemma 3.8. *For every subset* $W \subset U$, *we have the following.*

(1) $(U - W)^* = U - W_*$.

(2) $(U - W)_* = U - W^*$.

Corollary 3.9. *Intersection of two nano pre-closed sets is nano pre-closed.*
Proof *Let* $A,B \in NPF(U)$. *Then we have,* $Ncl[Nint(A \cap B)] = Ncl[Nint(A) \cap Nint(B)] \subset Ncl(Nint(A)) \cap Ncl(Nint(B)) \subset A \cap B$. *Thus,* $A \cap B$ *in* $NPF(U)$.

Theorem 3.10. *A subset of a space U is nano pre-open iff it is a nano pre-neighbourhood of each of its points.*
Proof: Let $G \subset U$ be a nano pre-open set. Then by definition it is clear that G is a nano pre-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is nano pre-open.
Conversely, suppose G is a nano pre-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in NPO(x)$ such that $S_x \subset G$. Then, $G = \bigcup_{x \in G} S_x$.

Since each S_x is nano pre-open it follows that G is nano pre-open by Lemma-3.4.

Lemma 3.11. *Let A be a set in a space U. A point* $x \in U$ *is in the nano pre-interior of A iff there is a* $G \in NPO(x)$ *such that* $G \subset A$.
Proof: Suppose $x \in NA_*$. By definition of NA_* there exists $G \in NPO(x)$ such that $x \in G$ and $G \subset A$. Hence there is $G \in NPO(x)$, such that $G \subset A$. Conversely, suppose $G \in NPO(x)$, such that $G \subset A$. Then $x \in G \subset NA_*$. Hence $x \in A_*$.

Now, we define the following.

Definition 3.12. *A point* $x \in U$ *is called a nano pre-interior point of* $A \subset U$ *if* $x \in NA_*$.

In view of this definition and Lemma 3.11, one can prove the following.

Lemma 3.13. *Let U be a space and* $A \subset X$, *and* $x \in X$. *Then x is a nano pre-interior point of A iff A is a nano pre-neighbourhood of x.*

Note 3.14. Since every nano open set is nano pre-open, every nano-interior point of a set $A \subset U$ is a nano pre-interior point of A. Thus, $Nint(A) \subset NA_*$. In general, $Nint(A) \neq NA_*$, which is shown by the following.

Example 3.15. Consider the set U = {a,b,c} equipped with the nano topology $\tau_R(X)$ = {ϕ,{b},U}. Then we obtain NPO(X) = {ϕ,{b},{a,b},{b,c},U}. Now, if we take A = {a,b} then Nint(A) = {b} and NA_* = {a,b}. This shows that Nint(A) ≠ NA_*.

Theorem 3.16. *Let U be a space and A ⊂ U. Then NA_* is the largest nano pre-open subset of U contained in A.*

Proof: To prove NA_* is the largest nano pre-open set contained in A. In other words, to show that NA_* contains any other nano pre-open set which is contained in A. Now, assume that U is any nano pre-open set with U ⊂ A. Let x ∈ U. Then by definition x ∈ U ⊂ A. Therefore A is a nano pre-neighbourhood of x ∈ U. This shows that x is a nano pre-interior point of A. Then x ∈ NA_* by Lemma 3.13 as x ∈ U implies x ∈ NA_*. Thus U ⊂ NA_* and NA_* is nano pre-open. Therefore NA_* contains every nano pre-open set X contained in A and hence NA_* is the largest nano pre-open set contained in A.

Theorem 3.17. *A is nano pre-open iff A = NA_*.*

Proof: Suppose A = NA_*. As NA_* is nano pre-open set, by hypothesis, A is nano pre-open. Next suppose that A is nano pre-open. Then A is a nano pre-open set contained in A. But NA_* is the largest nano pre-open set contained in A by Theorem 3.10. Therefore, A ⊂ NA_*. But NA_* ⊂ A always. Hence A = NA_*.

Lemma 3.18. *If A ⊂ B then NA_* ⊂ NB_*.*

Easy Proof is omitted.

Note 3.19. *NA_* = NB_* does not imply that A = B. This is shown by the following.*

Example 3.20. *Let X = {a,b,c} and $\tau_R(X)$ = {U,ϕ,{a},{b,c}}. Then it can be readily verified that $\tau_R(X)$ is a nano topology on U and, NPO(U) = $\tau_R(X)$. Take, A = {a} and B = {a,b}. Then, we obtain, NA_* = {a} = NB_*. But A ≠ B.*

Lemma 3.21. *Let A and B be subsets of U. Then,*

(1) $NA_* \cup NB_* \subset N(A \cup B)_*$.

(2) $N(A \cap B)_* \subset NA_* \cap NB_*$.

Proof follows by Lemma 3.18

In general, $N(A \cap B)_* \neq NA_* \cap NB_*$ as shown by the following.

Example 3.22. Let U = {a,b,c} and $\tau_R(X)$ = {U,ϕ,{a,b,c}}. Then it can be verified that $\tau_R(X)$ is a nano topology on U and, NPO(U) = {{a},{b},{c},{a,b},{a,c},{b,c},U,ϕ}. Take, A = {a} and B = {a,c}, then A ∩ B = {a}. Then, we have, NA_* = {a}, NB_* = {a,c} and $N(A \cap B)_*$ = ϕ. Thus, it follows that $NA_* \cap NB_*$ = {a} ≠ ϕ = $N(A \cap B)_*$.

Next, we define the nano pre-limit point of a subset of space U in the following.

Definition 3.23. *A point $x \in U$ is said to be a nano pre-limit point of A iff for each $X \in NPO(X)$, $X \cap (A - \{x\})$ 6= ϕ.*

Remark 3.24. Since every nano-open set is nano pre-open, it follows that every nano pre-limit point of A is a nano-limit point of A.

Definition 3.25. *The set of all nano pre-limit points of A is said to be the nano pre-derived set of A and in denoted by NPD(A).*

Note 3.26. By Remark 3.18 it follows that NPD(A) ⊂ ND(A), where ND(A) is the nano-derived set of A.

But in general, converse does not hold.

Example 3.27. Let U = {a,b,c,d} with the topology $\tau_R(X)$ = {ϕ,{b},{d},X}. Then, NPO(X) = {{ϕ,{b},{d},{b,d},{b,c,d},X}. For A = {a,c,d}, ND(A) = {a,c} and NPD(A) = {c}. Hence ND(A) ⊄ NPD(A).

Lemma 3.28. *A is nano pre-closed set iff it contains the set of its nano pre-limit points.*

Proof: By definition, U − A is nano pre-open set since A is nano pre-closed. Thus, A is nano pre-closed iff each point of (U −A) has a nano pre-neighbourhood contained in (U − A) iff no point of (U − A) is nano pre-limit point of A, or equivalently that A contains each of its nano pre-limit points.

Lemma 3.29. *Let A and B be subsets of a space U and A ⊂ B. Then, A ⊂ B implies NPD(A) ⊂ NPD(B).*

Proof: Let x ∈ U be a nano pre-limit point of A. Then by definition, there exists U ∈ NPO(U) such that U∩(A−{x}) ≠ ϕ and, hence it follows that X∩(B−{x}) ≠ ϕ, i.e., x is a nano pre-limit point of B. Thus, the nano pre-derived set of A is a subset of the nano pre-derived set of B.

In general, the converse does not hold in the above Lemma. This is shown by the following.

Example 3.30. Consider the space {U,τ_R(X)} as defined in the Example 3.20. Let A = {a,c} and B = {b,c}. Then, we obtain NPD(A) = {b} = NPD(B). But A ≠ B.

Theorem 3.31. *Let A and B be subsets of a space U. Then we have the following properties.*

(1) $NPD(\phi) \neq \phi$.
(2) $x \in NPD(A)$ *implies* $x \in NPD(A - \{x\})$.
(3) $NPD(A) \cup NPD(B) \subset NPD(A \cup B)$.
(4) $NPD(A \cap B) \subset NPD(A) \cap NPD(B)$.

Proof:

(1) It is obvious.

(2) Let $x \in NPD(A)$. Then x is a nano pre-limit point of A. That is, every nano pre-neighbourhood of x contains at least one point of A other than x. It means that every nano pre-neighbourbood of x contains at least one point other than x of $A - \{x\}$. Hence x is a nano pre-limit point of $A - \{x\}$ and therefore $x \in NPD(A - \{x\})$.

(3) and (4) follow by Lemma - 3.29.

Lemma 3.32. *Let U be a space and A be subset of U. Then $A \cup NPD(A)$ is a nano pre-closed set.*

Proof: Let $x \notin UNPD(A)$. This implies $x \notin A$ and $x \notin NPD(A)$. Since $x \notin NPD(A)$, there exists a nano pre-open neighbourhood N_x of x which contains no point of A other than x. But $x \notin A$. So N_x contains no point of A, which implies $N_x \subset U - A$. Again, N_x is a nano pre-open neighbourhood of each of its points. But as N_x does not contain any point of A, no point of N_x can be a nano pre-limit point of A. Therefore no point of N_x can belong to NPD(A). This implies that $N_x \subset X - NPD(A)$. Hence, it follows that $x \in N_x \subset (X - (A \cup NPD(A)))$.

Therefore, $A \cup NPD(A)$ is nano pre-closed.

Remark 3.33. Comparing the results of Lemma-3.32 and Lemma-3.7(i), one can easily write that, $A \cup NPD(A)$ is nano pre-closed iff $NA^* = A \cup NPD(A)$. It is obvious that $NA^* \subset Ncl(A)$.

The converse may be false as shown by the following.

Example 3.34. Let $U = \{a,b,c,d\}$ and $\tau_R(X) = \{\{c\},\{a,d\},\{a,c,d\}\}$. Then it can be easily verified that $\tau_R(X)$ is a nano topology on U and $NPO(U) = \{U,\phi,\{a\},\{c\},\{d\},\{a,c\},\{a,d\},\{b,d\},\{a,b,c\},\{a,c,d\},\{b,c,d\}\}$. Take, $A = \{b,d\}$. Then we have $Ncl(A) = \{a,b,d\}$ and $NA^* = \{b,d\}$ which shows that $Ncl(A) \not\subset NA^*$.

Note 3.35. If $NA^* = NB^*$. Then it does not imply $A = B$.

This is shown by the following.

Example 3.36. Let $U = \{a,b,c,d\}$ and $\tau_R(X) = \{c,\{a,d\},\{a,c,d\}\}$ be a nano topology on U. Then we obtain $NPO(U) = \{U,\phi,\{a\},\{c\},\{d\},\{a,c\},\{a,d\},\{b,d\}, \{a,b,c\},\{a,c,d\},\{b,c,d\}\}$. Then $NPF(U) = \{\{b\},\{b,c\},\{a,b,d\}, U,\phi\}$. Take $A = \{c\}$ and $B = \{b,c\}$. Then $NA^* = \{b,c\} = NB^*$.

Hence, It follows that $A \neq B$ even though $NA^* = NB^*$.

Theorem 3.37. *$NA^* = A - NPD(U - A)$.*

Proof: Let $x \in A - NPD(U - A)$. Then $x \in A$ and $x \notin NPD(U - A)$. Since $x \notin NPD(U-A)$, there exists $U \in N\text{-}PO(x)$ with $U \cap (U-A) = \phi$. Hence, $x \in X \subset A$, which Implies $x \in NA_*$. Conversely, let $x \in NA_*$. Then $x \notin NPD(U - A)$, for NA_* is nano pre-open neighbourhood of x and, $NA_* \cap (U-A) = \phi$. Also, $NA_* \subset A$. This shows that $x \in A$. Hence $NA_* = A - NPD(U - A)$.

Remark 3.38. *Using Lemma 3.8, one can write that $NA_* = U - (U - A)^*$*

Theorem 3.39. *For subsets A,B ⊂ U. The following hold.*

(1) $NA^* \cup NB^* \subset (NA \cup NB)^*$.

(2) $(NA \cap NB)^* \subset NA^* \cap NB^*$.

Proof:

(1) Since $A \subset A \cup B$ and $B \subset A \cup B$. Then by Lemma 2.8(ii), we obtain that $NA^* \subset (NA \cup NB)^*$ and $NB^* \subset (NA \cup NB)^*$. It follows that $NA^* \cup NB^* \subset (NA \cup NB)^*$.

(2) $(NA \cap NB)^*$subset NA^* and $(A \cap B)^* \subset NB^*$. Hence It follows that $(A \cap B)^* \subset NA^* \cap NB^*$.

In general, the equality does not hold in the above Theorem. This can be shown by the following.

Example 3.40. Let U = {a,b,c,d} and $\tau_R(X)$ = {{c},{a,d},{a,c,d},U,ϕ}. Take A = {c} and B = {d} then $A \cup B$ = {c,d}. Then NA^* = {b,c}, NB^* = {d} and $(A \cup B)^*$ = {b,c,d}. It follows that $NA^* \cup NB^* = (A \cup B)^*$.

Example 3.41. Let U = {a,b,c,d} and $\tau_R(X)$ = {{c},{a,d},{a,c,d},U,ϕ}. Take, A = {a} and B = {c,d} then $A \cap B$ = ϕ. Then NA^* = {b} and NB^* = {b,c,d} and $(A \cap B)^*$ = {b}. Thus $NA^* \cap NB^* = (NA \cap NB)^*$.

Next we define nano pre-frontier of subset of a space.

Definition 3.42. *$NA^* - NA_*$ is said to be the nano pre-frontier of $A \subset U$ and is denoted by NPfr(A). It is obvious that NPfr(A) ⊂ Nfr(A), the nano-frontier of A.*

But in general the converse may not be true.

Example 3.43. Let U = {a,b,c,d} and $\tau_R(X)$ = {{c},{a,d},{a,c,d},U,ϕ}. If A = {a,b,d}. Then, Nint(A) = {a,d}, Ncl(A) = {a,b,d}, NA^* = {a,b,d} and NA_* = {b,d}. This shows that Nfr(A) ⊄ NPfr(A).

Lemma 3.44. *For a subset A of a space U,*

(1) $NA^* = NA_* \cup NPfr(A)$

(2) $NA_* \cap NPfr(A) = \phi$ *and* (3) $NPfr(A) = NA^* \cap (U - A)^*$.

Proof: By definition of NPfr(A), we have

(1) $NA_* \cup NPfr(A) = NA_* \cup (NA^* - NA_*) = NA^*$.

(2) $NA_* \cap NPfr(A) = NA_* \cup (NA^* - NA_*) = \phi$.

(3) $NPfr(A) = NA^* - NA_* = NA^* \cap (U - NA_*) = NA^* \cap (U - NA)^*$ by Lemma-3.8(i).

Lemma 3.45. *NPfr(A) is pre-closed.*

Proof: By Lemma 3.44, $NPfr(A) = NA^* \cap (U - NA)^*$, which is nano pre-closed by Corollary 3.9.

Now, we define the following.

Definition 3.46. *A subset $A \subset U$ is called nano pre-regular if it is both nano pre-open and nano pre-closed set. The family of all nano pre-regular sets of U is denoted by NPR(U). The complement of a nano pre-regular set is also a nano pre-regular.*

Next, we prove the following.

Theorem 3.47. *$NPfr(A) = \phi$ iff $A \in NPR(U)$.*
Proof: Let $A \in NPR(U)$. Then $A \in NPO(U)$ and $A \in NPF(U)$. Now, using results of Lemma 3.7 and Theorem 3.17 it follows that $NPfr(A) = \phi$. Conversely, let $NPfr(A) = \phi$. Then we show that $A \in NPR(U)$. Since by hypothesis, $NA^* - NA_* = \phi$. We have $NA^* = NA_*$. But, $NA_* \subset A \subset NA^*$. Therefore, it follows that $A = NA_* = NA^*$ which means $A \in N - PR(U)$.

Theorem 3.48. *Let A be subset of U. Then, the following hold.*

(1) $NPfr(A) = NPfr(U - A)$.
(2) $A \in NPO(U)$ *iff* $NPfr(A) \subset U - A$. *i.e.,* $A \cap NPfr(A) = \phi$.
(3) $A \in NPF(U)$ *iff* $NPfr(A) \subset A$.

Proof:

(1) We have, $NPfr(U - A) = (U - NA)^* \cap (U - (U - NA))^* = (U - NA)^* \cap NA^* = NPfr(A)$ by Lemma 3.44(iii).

(2) Assume $A \in NPO(U)$. By definition, we have $NPfr(A) = NA^* - NA_* = NA^* - A$. Since $A \in NPO(U)$. Then, $A \cap NPfr(A) = A \cap (NA^* - A) = NA^* \cap (X - A) \cap A = \phi$. Conversely, if $A \cap NPfr(A) = \phi$. Then, $A \cap NA^* \cap (U - NA_*) = \phi$ implies $A \cap (U - NA_*) = \phi$ as $A \subset NA^*$. Thus, $A \subset U - (U - NA_*) = NA_*$, but on the other hand $NA_* \subset A$. It follows that $A = NA_*$, which implies $A \in NPO(U)$.

(3) Assume $A \in NPF(U)$. Then, we have $U - A \in NPO(U)$. Then by (ii), $NPfr(U-A) \cap (U-A) = \phi$. But, by (i), $NPfr(U-A) = NPfr(A)$. Hence $NPfr(A) \cap (U - A) = \phi$. This shows that $NPfr(A) \subset A$. Conversely, if $NPfr(A) \subset A$, then $NA^* - NA_* \subset A$, which implies $NA_* \cup (NA^* - NA_*) \subset A \cup NA_* = A$, which implies $NA^* \subset A$ by 3.44(i) But $A \subset NA^*$. It follows that $A = NA^*$. Hence $A \in NPF(U)$.

Note 3.49. Let A and B be subsets of space U. Then $A \subset B$ does not imply that either $NPfr(A) \subset NPfr(B)$ or $NPfr(B) \subset NPfr(A)$.

This can be verified by the following.

Example 3.50. Let $U = \{a,b,c,d\}$ and $\tau_R(X) = \{\{c\},\{a,d\},\{a,c,d\},U,\phi\}$. Then, case(1) : Take $A = \{d\}$ and $B = \{a,c,d\}$. Then $A \subset B$. Also $NA^* = \{d\}$, $NA_* = \{d\}$ and $NPfr(A) = \{\phi\}$. $NB^* = \{U\}$, $NB_* = \{a,c,d\}$ and $pfr(B) = \{b\}$. This shows that NPfr(A) 6⊂ NPfr(B).
Case-(2): If $A = \{d\}$ and $B = \{c,d\}$ then $A \subset B$. We obtain that $NPfr(A) = \{\phi\}$ and $NPfr(B) = \{d\}$. Thus, NPfr(B) 6⊂ NPfr(A).

Theorem 3.51. *If $A \in NPO(U) \cup NPF(U)$, then $NPfr(A) = NPfr(NPfr(A))$.*
Proof follows by Lemma 3.44(iii), Lemma 3.45 and Lemma 3.48(ii)(iii).

Corollary 3.52. *For every* $A \subset U$, $NPfr(NPfr(NPfr(A))) = NPfr(NPfr(A))$.
Proof is obvious.

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