Received: 25 October 2020 | Accepted: 29 December 2020
DOI: 10.1002/num.22772

RESEARCH ARTICLE

WILEY

New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$

V. Vijayakumar[1] | **Chokkalingam Ravichandran**[2] | **Kottakkaran Sooppy Nisar**[3] | **Kishor D. Kucche**[4]

[1]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India
[2]Post Graduate and Research Department of Mathematics, Kongunadu Arts and Science College(Autonomous), Coimbatore, Tamil Nadu, India
[3]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia
[4]Department of Mathematics, Shivaji University, Kolhapur, Maharashtra, India

Correspondence
V. Vijayakumar, Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India.
Email: vijaysarovel@gmail.com
Kottakkaran S. Nisar, Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia.
Email: n.sooppy@psau.edu.sa

Abstract

In our article, we are primarily concentrating on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential inclusions of order $1 < r < 2$. By applying the results and ideas belongs to the cosine function of operators, fractional calculus and fixed point approach, the main results are established. Initially, we establish the approximate controllability of the considered fractional system, then continue to examine the system with the concept of nonlocal conditions. In the end, we present an example to demonstrate the theory.

KEYWORDS

approximate controllability, fractional derivative, integro-differential system, nonlocal conditions, Sobolev-type system

1 | INTRODUCTION

Fractional calculus, as a significant area of Mathematics, was initiated in 1695. It was nearly simultaneously as classical calculus. Recently, the ideas about fractional calculus became effectively applied to different regions, and the investigators progressively found that the fractional calculus can well portray several non-local events in the areas of natural science and architecture. Presently, the mainstream areas of fractional calculus containing rheology, liquid stream, diffusive transport likened to dispersion, dynamical cycles in self-comparable and porous structures, viscoelasticity, and, optics, etc. The effective utilization of the fractional systems in these areas has incited several researchers to contemplate their mathematical estimate strategies, as the diagnostic arrangements are commonly hard to get. For interesting results related to fractional dynamical systems, the readers may refer to the books [1–4] and the articles [5–19].

Control theory is a significant region of utilization situated in Mathematics which manages the structure and examination of control frameworks. Recently, controllability concerns for different sorts of nonlinear dynamical frameworks in infinite dimensional spaces became examined in numerous research articles by utilizing various types of techniques. A broad list of these distributions can be discovered in [6, 9, 11–16, 18–24, 43, 44–50]. The studies about existence and controllability related to fractional evolution system of order $1 < \alpha < 2$ attracted many researchers and one can review the articles [10, 17, 25–33]. The study related to approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential inclusions having order $1 < r < 2$ have not been discussed yet and it gives the additional motivation for writing this article.

Motivated by the theory developed in the works mentioned previously, our objective in this article is to discuss the approximate controllability of Sobolev type Volterra-Fredholm integro-differential inclusions of fractional order $r \in (1, 2)$ of the form

$$ {}^{C}D^{r}_{\varrho}(Jz(\varrho)) \in Az(\varrho) + \mathscr{B}x(\varrho) + E\left(\varrho, z(\varrho), \int_0^{\varrho} e(\varrho, s, z(s))\,ds, \int_0^{c} h(\varrho, s, z(s))\,ds\right), \quad \varrho \in V = [0, c], \tag{1.1} $$

$$ z(0) = z_0, \quad z'(0) = z_1 \in \mathscr{Y}, \tag{1.2} $$

where $z(\cdot)$ takes values in $\mathscr{Y}$ and $\mathscr{Y}$ is a Hilbert space; ${}^{C}D^{r}_{\varrho}$ represents fractional derivative in Caputo sense of order $r \in (1, 2)$, and the control function $x(\cdot) \in L^2(V, \mathscr{U})$, a Hilbert space of admissible control functions. A is the infinitesimal generator of a C_0- cosine family $\{C(\varrho)\}_{\varrho \geq 0}$ on $\mathscr{Y}$. $\mathscr{B}$ is a bounded linear operator from $\mathscr{U} \to \mathscr{Y}$. Here $D = \{(\varrho, s) \in V \times V : s \leq \varrho\}$, $e, h : D \times \mathscr{Y} \to \mathscr{Y}$ are continuous, and $E : V \times \mathscr{Y} \times \mathscr{Y} \times \mathscr{Y} \to 2^{\mathscr{Y}} \setminus \{\emptyset\}$ satisfying some conditions.

The arrangement of our article as follows: In Section 2, we recollect several fundamental definitions and few well-known results belong to fractional calculus and multivalued maps. In Section 3, we present the approximate controllability of (1.1)–(1.2) and in Section 4, we examine the system with concept of nonlocal conditions. In Section 5, the theory is validated with suitable example.

2 | BASIC FACTS

We now recollect few well-known results and definitions for proving our main results of this article.

Assume that $C(V, \mathscr{Y}) : V \to \mathscr{Y}$ be the Hilbert space of continuous functions along with $\|z\| = \sup_{\varrho \in V} \|z(\varrho)\|$, $z \in C(V, \mathscr{Y})$. Denote $D(A)$, $R(A)$ be the domain and range of A. $\rho(A)$ stands for the resolvent set of A and we define the resolvent as follows:

$$ R(\Lambda, A) = (\Lambda I - A)^{-1} \in L_c(\mathscr{Y}). $$

A measurable function $g : V \to \mathscr{Y}$ is Bochner's integrable provided $\|g\|$ is Lebesgue. Let $L^p(V, \mathscr{Y})$ $(p \geq 1)$ be the Hilbert space of measurable functions endowed along with

$$ \|g\|_p = \left(\int_V \|g(\varrho)\|^p d\varrho\right)^{\frac{1}{p}}. $$

We now present the linear operators $A : D(A) \subset \mathcal{Y} \to \mathcal{Y}$ and $J : D(A) \subset \mathcal{Y} \to \mathcal{Y}$ satisfy the following properties established in [34]:

(E₁) $D(J) \subset D(A)$ and J is bijective.

(E₂) The operators A and J are closed linear operators.

(E₃) $J^{-1} : \mathcal{Y} \to D(J)$ is continuous.

Additionally, because of **(E₁)** and **(E₂)**, J^{-1} is closed, by **(E₃)** and by referring the closed graph theorem, $AJ^{-1} : \mathcal{Y} \to \mathcal{Y}$ is bounded. Denote $\|J^{-1}\| = \widetilde{J}_1$ and $\|J\| = \widetilde{J}_2$.

By referring [1], we introduce definitions and remarks belongs to fractional calculus.

Definition 2.1 The integral of fractional order ν for the function $f : [0, \infty) \to \mathbb{R}$ with lower limit zero is given by

$$I^{\nu} f(\varrho) = \frac{1}{\Gamma(\nu)} \int_0^{\varrho} \frac{f(\iota)}{(\varrho - \iota)^{1-\nu}} d\iota, \quad \varrho > 0, \ \nu \in \mathbb{R}^+.$$

Definition 2.2 The R-L derivative of order ν for the function $f : [0, \infty) \to \mathbb{R}$ having lower limit zero is given by

$${}^{L}D^{\nu} f(\varrho) = \frac{1}{\Gamma(n - \nu)} \frac{d^n}{d\varrho^n} \int_0^{\varrho} \frac{f^{(n)}(\iota)}{(\varrho - \iota)^{\nu+1-n}} d\iota, \quad \varrho > 0, \ n - 1 < \nu < n, \quad \nu \in \mathbb{R}^+.$$

Definition 2.3 The Caputo derivative of order ν for the function f having lower limit zero is given by

$${}^{C}D^{\nu} f(\varrho) = {}^{L}D^{\nu} \left(f(\varrho) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} \varrho^j \right), \quad \varrho > 0, \ n - 1 < \nu < n, \quad \nu \in \mathbb{R}^+.$$

Remark 2.4 (1) If $f(\varrho) \in C^n[0, \infty)$, then

$${}^{C}D^{\nu} f(\varrho) = \frac{1}{\Gamma(n - \nu)} \int_0^{\varrho} \frac{f^{(n)}(\iota)}{(\varrho - \iota)^{\nu+1-n}} d\iota = I^{n-\nu} f^{(n)}(\varrho), \quad \varrho > 0, \ n - 1 < \nu < n.$$

(2) The above integrals are considered in Bochner's sense if the function g is abstract with values belonged to $\mathcal{Y}$.

(3) The Caputo derivative of a constant function is equal to zero.

Definition 2.5 [35] The operator $\{C(\varrho)\}_{\varrho \in \mathbb{R}} : \mathcal{Y} \to \mathcal{Y}$ is called a strongly continuous cosine family if and only if

(a) $C(\iota + \varrho) + C(\iota - \varrho) = 2C(\iota)C(\varrho)$, for all $\iota, \varrho \in \mathbb{R}$;

(b) $C(\varrho)z$ is strongly continuous on $\mathbb{R}$ for every $z \in \mathcal{Y}$;

(c) $C(0) = I$.

Assume that the sine family associated with $\{C(\varrho)\}_{\varrho \in \mathbb{R}}$ is $\{S(\varrho)\}_{\varrho \in \mathbb{R}}$, where

$$S(\varrho) z = \int_0^{\varrho} C(\iota) z d\iota, \quad z \in \mathcal{Y}, \quad \varrho \in \mathbb{R}. \tag{2.1}$$

Further, if

$$\left(Az = \frac{d^2}{d\varrho^2} C(\varrho) z \right|_{\varrho=0}, \quad \textit{for all } z \in D(A),$$

where

$$D(A) = \left\{ z \in \mathcal{Y} : C(\varrho) z \in C^2(\mathbb{R}, \mathcal{Y}) \right\}.$$

Determine

$$G = \left\{ z \in \mathscr{Y} \; : \; C(\varrho) z \in C^1(\mathbb{R}, \mathscr{Y}) \right\}.$$

Clearly, A is a closed dense operator in $\mathscr{Y}$, there exists a constant $P \geq 1$ if $\|N(\varrho)\|_{L_c(\mathscr{Y})} \leq P$, where $\varrho \geq 0$. Let us fix $a = \frac{r}{2}$ for $r \in (1, 2)$ which is discussed in [25, 33].

Definition 2.6 The upper semicontinuous (u.s.c) operator $\mathcal{K}$ on $\mathscr{Y}$ provided for all $z_0 \in \mathscr{Y}$ and $\mathcal{K}(z_0)$ is a nonempty closed subset of $\mathscr{Y}$, for each open set $\mathscr{C}$ of $\mathscr{Y}$ including $\mathcal{K}(z_0)$, there exists an open neighborhood $\mathcal{W}$ of z_0 with

$$\mathcal{K}(\mathcal{W}) \subseteq \mathscr{C}.$$

Definition 2.7 $\mathcal{K}$ is completely continuous if $\mathcal{K}(\mathscr{C})$ is relatively compact for each bounded subset $\mathscr{C}$ of $\mathscr{Y}$. If $\mathcal{K}$ is completely continuous and nonempty, then $\mathcal{K}$ is u.s.c., if and only if $\mathcal{K}$ has a closed graph. $\mathcal{K}$ has a fixed point such that there exists $z \in \mathscr{Y}$ such that $z \in \mathcal{K}(z)$.

The fractional system (1.1)–(1.2) is similar to the following system

$$\begin{aligned} z(\varrho) = {} & J^{-1} z_0 J + J^{-1} z_1 J \varrho \\ & + \frac{1}{\Gamma(r)} \int_0^{\varrho} (\varrho - \iota)^{r-1} J^{-1} \left[A z(\iota) + E\left(\iota, z(\iota), \int_0^{\iota} e(\iota, s, z(s))\, ds, \int_0^{c} h(\iota, s, z(s))\, ds \right) \right] d\iota \\ & + \frac{1}{\Gamma(r)} \int_0^{\varrho} (\varrho - \iota)^{r-1} J^{-1} \mathscr{Y} x(\iota)\, d\iota. \end{aligned} \tag{2.2}$$

In view [36] and using Laplace transform, we introduce the solution of (1.1)–(1.2).

Definition 2.8 [36] A function $z \in C = C(V, \mathscr{Y})$ is called a mild solution of (1.1)–(1.2) if $z(0) = z_0$, $z'(0) = z_1$, $x(\cdot) \in L^2(V, \mathscr{U})$ and there exists $g \in L^1(V, \mathscr{Y})$ such that $g(\varrho) \in E\left(\varrho, z(\varrho), \int_0^{\varrho} e(\varrho, s, z(s))\, ds, \int_0^{c} h(\varrho, s, z(s))\, ds\right)$ on a.e. $\varrho \in V$ and

$$\begin{aligned} z(\varrho) = {} & J^{-1} M_a(\varrho) J z_0 + J^{-1} W_a(\varrho) J z_1 + \int_0^{\varrho} (\varrho - \iota)^{a-1} J^{-1} Q_a(\varrho - \iota) \mathscr{B} x(\iota)\, d\iota \\ & + \int_0^{\varrho} (\varrho - \iota)^{a-1} J^{-1} Q_a(\varrho - \iota) g(\iota)\, d\iota, \end{aligned} \tag{2.3}$$

where

$$M_a(\varrho) = \int_0^{\infty} S_a(\rho) C(\varrho^a \rho)\, d\rho, \quad W_a(\varrho) = \int_0^{\varrho} M_a(\iota)\, d\iota, \quad Q_a(\varrho) = \int_0^{\infty} a \rho S_a(\rho) S(\varrho^a \rho)\, d\rho,$$

$$S_a(\rho) = \frac{1}{a} \rho^{-1-\frac{1}{a}} \zeta_a\left(\rho^{-\frac{1}{a}}\right), \quad \zeta_a(\rho) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \rho^{-na-1} \frac{\Gamma(na+1)}{n!} \sin(n\pi a), \quad \rho \in (0, \infty),$$

where $S_a(\cdot)$ is a probability density function defined on $(0, \infty)$ such that

$$S_a(\rho) \geq 0 \;\; where \;\; \rho \in (0, \infty) \;\; also \;\; \int_0^{\infty} S_a(\rho)\, d\rho = 1.$$

Remark 2.9 Clearly for $n \in [0, 1]$

$$\int_0^{\infty} \rho^n S_a(\rho)\, d\rho = \int_0^{\infty} \rho^{-an} \zeta(\rho)\, d\rho = \frac{\Gamma(1+n)}{\Gamma(1+an)}.$$

Lemma 2.10 [25]. *$M_a(\varrho)$, $W_a(\varrho)$ and $Q_a(\varrho)$ satisfy the following properties:*
(a) for all $\varrho \geq 0$, *$M_a(\varrho)$, $W_a(\varrho)$ and $Q_a(\varrho)$ are linear and bounded;*
(b) $\forall\ z \in \mathscr{Y}$ *and for all* $\varrho \geq 0$, *we have*

$$\|M_a(\rho) z\| \leq P\|z\|, \quad \|W_a(\rho) z\| \leq P\|z\|\rho, \quad \|Q_a(\rho) z\| \leq \frac{P}{\Gamma(2a)}\|z\|\rho^a;$$

(c) $\{M_a(\varrho), \varrho \geq 0\}$, $\{W_a(\varrho), \varrho \geq 0\}$ *and* $\{\varrho^{a-1}Q_a(\varrho), \varrho \geq 0\}$ *are strongly continuous.*

Lemma 2.11 [35].
(i) *If* $z \in G$, *subsequently* $S(\varrho)z \in D(A)$, *also;* $\frac{d}{d\varrho}C(\varrho)z = AS(\varrho)z$;
(ii) There exists $P \geq 1$, $\omega \geq 0$ such that $\|C(\varrho)\|_{L_c(\mathscr{Y})} \leq Pe^{\omega|\rho|}$, for all $\varrho \in \mathbb{R}$;
(iii) $\|S(\varrho + \hbar) - S(\varrho)\|_{L_c(\mathscr{Y})} \leq P \mid \int_{\varrho}^{\varrho+\hbar} e^{\omega|\imath|} d\imath \mid$ *for all* $\varrho + \hbar, \varrho \in \mathbb{R}$.

Lemma 2.12 *Assume that* $\{C(\varrho)\}_{\varrho \in \mathbb{R}}$ *on* $\mathscr{Y}$, *then*

$$\lim_{\varrho \to 0} \frac{1}{\varrho} S(\varrho) z = z$$

for some $z \in \mathscr{Y}$.

Lemma 2.13 [35] *Suppose* $\{C(\varrho)\}_{\varrho \in \mathbb{R}}$ *on* $\mathscr{Y}$ *satisfying* $\|C(\varrho)\|_{L_c(\mathscr{Y})} \leq Pe^{\omega|\varrho|}$, $\varrho \in \mathbb{R}$. *Subsequently for* $Re\Lambda > \omega$, $\Lambda^2 \in \rho(A)$. *Additionally*

$$\Lambda R\left(\Lambda^2; A\right) z = \int_0^\infty e^{-\Lambda\varrho} C(\varrho) z d\varrho, \quad R\left(\Lambda^2; A\right) z = \int_0^\infty e^{-\Lambda\varrho} S(\varrho) z d\varrho, \quad \text{for } z \in \mathscr{Y}.$$

We present the essential operators and some useful results as follows:

$$\Gamma_0^c = \int_0^c (c - \imath)^{a-1} J^{-1} Q_a(c - \imath) \mathscr{B}\mathscr{B}^* J^{-1} Q_a^*(c - \imath) d\imath : \mathscr{Y} \to \mathscr{Y},$$
$$R\left(\mu, \Gamma_0^c\right) = \left(\mu I + \Gamma_0^c\right)^{-1} : \mathscr{Y} \to \mathscr{Y},$$

where $\mathscr{B}^*$, $Q_a^*(c)$ stand for adjoints $\mathscr{B}$ and $Q_a(c)$. Now, we come to an end that Γ_0^c is bounded.

We start with the following assumption:

$\mathbf{H_0}$ $\mu R\left(\mu, \Gamma_0^c\right) \to 0$ as $\mu \to 0^+$ in the strong operator topology.

In view of [22], $\mathbf{H_0}$ holds if and only if the linear fractional system

$$\begin{cases} {}^C D_{0+}^r (Jz(\varrho)) \in Az(\varrho) + (\mathscr{B}x)(\varrho), & \varrho \in V, \\ z(0) = z_0, \ z'(0) = z_1 \in \mathscr{Y}, \end{cases} \tag{2.4}$$

is approximately controllable on V.

Lemma 2.14 [36]. *Suppose* $BCC(\mathscr{Y})$ *be the set of all nonempty, bounded, closed and convex subset of* $\mathscr{Y}$, V *be a compact real interval. Let* E *be a multivalued map satisfying* $E : V \times \mathscr{Y} \to BCC(\mathscr{Y})$ *is measurable to* ϱ *for every fixed* $z \in \mathscr{Y}$, *u.s.c. to* z *for every* $\varrho \in V$, *and for every* $z \in C$,

$$S_{E,z} = \left\{ g \in L^1(V, \mathscr{Y}) : g(\varrho) \in E\left(\varrho, z(\varrho), \int_0^\rho e(\varrho, s, z(s)) ds, \int_0^c h(\varrho, s, z(s)) ds\right), \varrho \in V \right\}$$

is nonempty. Suppose $\mathcal{M}$ be linear continuous from $L^1(V, \mathscr{Y}) \to C$, then

$$\mathcal{M} \circ S_E : C \to BCC(C), \ z \to (\mathcal{M} \circ S_E)(z) = \mathcal{M}\left(S_{E,z}\right),$$

is a closed graph operator in $C \to C$.

Lemma 2.15 *Suppose H be a closed, bounded and convex nonempty subset H of $\mathscr{Y}$. Assume $\mathcal{D} : H \to 2^{\mathscr{Y}} \setminus \{\emptyset\}$ is upper semicontinuous with closed, convex values such that $\mathcal{D}(H) \subset H$ and $\mathcal{D}(H)$ is compact, then $\mathcal{D}$ has a fixed point.*

3 | RESULTS ON APPROXIMATE CONTROLLABILITY

We here particularly focusing on approximate controllability of (1.1)–(1.2). Let us introduce the important assumptions to discuss the primary theorems.

H$_1$ For $\varrho \geq 0$, $\{C(\varrho)\}$ is compact.

H$_2$ The function $E : V \times \mathscr{Y} \times \mathscr{Y} \times \mathscr{Y} \to BCC(\mathscr{Y})$ is measurable to ϱ, for all $z \in \mathscr{Y}$, u.s.c. to z, for all $\varrho \in V$, and for all $z \in C$

$$S_{E,z} = \left\{ g \in L^1(V, \mathscr{Y}) : g(\varrho) \in E\Big(\varrho, z(\varrho), \int_0^\varrho e(\varrho, s, z(s))\, ds, \int_0^c h\big(\varrho, s, z(s)\big) ds\Big), \varrho \in V \right\},$$

is nonempty.

H$_3$ For each $(\varrho, s) \in D$, the functions $e(\varrho, s, \cdot), h(\varrho, s, \cdot) : \mathscr{Y} \to \mathscr{Y}$ are continuous and for all $z \in \mathscr{Y}$, $e(\cdot, \cdot, z), h(\cdot, \cdot, z) : D \to \mathscr{Y}$ are strongly measurable.

H$_4$ For $\alpha > 0$, $z \in C$ along with $\|z\|_C \leq \alpha$, there exists $v \in (0, b)$ and $L_{g,\alpha}(\cdot) \in L^{\frac{1}{v}}\left(V, \mathbb{R}^+\right)$ such that

$$\sup \left\{ \|g\| : g(\varrho) \in E\Big(\varrho, z(\varrho), \int_0^\varrho e(\varrho, s, z(s))\, ds, \int_0^c h\big(\varrho, s, z(s)\big) ds\Big) \right\} \leq L_{g,\alpha}(\varrho),$$

for almost everywhere $\varrho \in V$.

H$_5$ The function $\imath \to (\varrho - \imath)^{a-1} L_{g,\alpha}(\imath) \in L^1(V, \mathbb{R}^+)$ and there exists a constant $\lambda > 0$ such that

$$\liminf_{\alpha \to \infty} \frac{\int_0^\varrho (\varrho - \imath)^{a-1} L_{g,\alpha}(\imath)\, d\imath}{\alpha} = \lambda < +\infty.$$

For discussing the approximate controllability of (1.1)–(1.2), if for all $\mu > 0$, there exists $z(\cdot) \in C$ such that

$$\begin{aligned} z(\varrho) = {} & J^{-1} M_a(\varrho) J z_0 + J^{-1} W_a(\varrho) J z_1 + \int_0^\varrho (\varrho - \imath)^{a-1} J^{-1} Q_a(\varrho - \imath)\, g(\imath)\, d\imath \\ & + \int_0^\varrho (\varrho - \imath)^{a-1} J^{-1} Q_a(\varrho - \imath)\, \mathscr{B} x(\imath)\, d\imath, \quad g \in S_{E,z}, \end{aligned} \tag{3.1}$$

$$x(\varrho) = \mathscr{B}^* J^{-1} Q_a^*(c - \varrho)\, R\left(\mu, \Gamma_0^c\right) q(z(\cdot)), \tag{3.2}$$

where

$$q(z(\cdot)) = z_c - J^{-1} M_a(c) J z_0 - J^{-1} W_a(c) J z_1 - \int_0^c (c - \imath)^{a-1} J^{-1} Q_a(c - \imath)\, g(\imath)\, d\imath.$$

Theorem 3.1 *Assume* **H$_0$–H$_5$** *holds, subsequently* (1.1)–(1.2) *has a solution on V provided*

$$\frac{P\widetilde{J}_1}{\Gamma(2a)} \left[1 + \frac{1}{\mu} \left(\frac{P\widetilde{J}_1 P_B}{\Gamma(2a)} \right)^2 \frac{c^{2a}}{2a} \right] \lambda < 1, \tag{3.3}$$

where $P_B = \|\mathscr{B}\|$.

Proof. The fundamental point is to discover conditions for solvability of (3.1) and (3.2) for $\mu > 0$. Now, we prove $\mathcal{R} : C \to 2^C$ determined by

$$\mathcal{R}(z) = \left\{ z \in C; m(\varrho) = J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)g(\imath)d\imath \right.$$
$$\left. + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)\mathscr{B}x(\imath)d\imath, g \in S_{E,z} \right\},$$

has a fixed point. We subdivide our proof for simplicity:

Step 1: For all $\mu > 0$, $\mathcal{R}(z)$ is convex for all $z \in C$. Suppose $m_1, m_2 \in C$, then there exists g_1, $g_2 \in S_{E,z}$ such that $\varrho \in V$, we have

$$m_i(\varrho) = J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)g_i(\imath)d\imath$$
$$+ \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c - \varrho)R\left(\mu, \Gamma_0^c\right)\left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1\right.$$
$$\left. - \int_0^{c} (c - \varphi)^{a-1}J^{-1}Q_a(c - \varphi)g_i(\varphi)d\varphi\right](\imath)d\imath, i = 1, 2.$$

Let $0 \le \beta \le 1$, then for each $\rho \in V$ we have

$$(\beta m_1 + (1 - \beta)m_2)(\varrho) = J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)\left[\beta g_1(\imath)\right.$$
$$\left. + (1 - \beta)g_2(\imath)\right]d\imath + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c - \varrho)R\left(\mu, \Gamma_0^c\right)\left[z_c - J^{-1}M_a(c)Jz_0\right.$$
$$\left. - J^{-1}W_a(c)Jz_1 - \int_0^{c} (c - \varphi)^{a-1}J^{-1}Q_a(c - \varphi)\left[\beta g_1(\varphi) + (1 - \beta)g_2(\varphi)\right]d\varphi\right](\imath)d\imath.$$

Since $S_{E,z}$ is convex, $\beta m_1 + (1 - \beta)m_2 \in S_{E,z}$. Hence $\beta m_1 + (1 - \beta)m_2 \in \mathcal{R}(z)$.

Step 2: On the space C, consider $\mathcal{Q}_\alpha = \{z \in C : \|z\|_C \le \alpha, 0 \le \varrho \le c\}$, $\alpha > 0$. Clearly, $\mathcal{Q}_\alpha$ is bounded, closed and convex set of C. For $\mu > 0$, our property is that there exists $\alpha > 0$ such that

$$\mathcal{R}(\mathcal{Q}_\alpha) \subset \mathcal{Q}_\alpha.$$

If not, then for all $\alpha > 0$, there exists $z^\alpha \in \mathcal{Q}_\alpha$, but $\mathcal{R}(z^\alpha) \notin \mathcal{Q}_\alpha$, i.e.,

$$\|\mathcal{R}(z^\alpha)\|_C \equiv \sup\{\|m^\alpha\|_C : m^\alpha \in (\mathcal{R}z^\alpha)\} > \alpha$$

and

$$m^\alpha(\varrho) = J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)g^\alpha(\imath)d\imath$$
$$+ \int_0^{\varrho} (\varrho - \imath)^{a-1}J^{-1}Q_a(\varrho - \imath)\mathscr{B}x^\alpha(\imath)d\imath,$$

for some $g^\alpha \in S_{E,z^\alpha}$.

Using assumptions $\mathbf{H_3}$ and Lemma 2.10, we have

$$\|x^\alpha(\rho)\| = \|\mathscr{B}^*J^{-1}Q_a^*(c - \varrho)R\left(\mu, \Gamma_0^c\right)q(z(\cdot))\|$$
$$= \|\mathscr{B}^*J^{-1}Q_a^*(c - \varrho)R\left(\mu, \Gamma_0^c\right)\left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1\right.$$
$$\left. - \int_0^{c} (c - \imath)^{a-1}J^{-1}Q_a(c - \imath)g^\alpha(\imath)d\imath\right]\|$$

$$\leq \|J^{-1}\|\|\mathscr{B}^*\|\|Q_a^*(c-\varrho)\|\|R\left(\mu,\Gamma_0^c\right)\|\|\Big[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1$$
$$- \int_0^c (c-\iota)^{a-1}J^{-1}Q_a(c-\iota)g^\alpha(\iota)d\iota\Big]\|$$
$$\leq P_B\widetilde{J}_1\left(\frac{P}{\Gamma(2a)}\right)\frac{1}{\mu}\Big[\|z_c\| + \|J^{-1}M_a(c)Jz_0\| + \|J^{-1}W_a(c)Jz_1\|$$
$$+ \int_0^c (c-\iota)^{a-1}\|J^{-1}Q_a(c-\iota)g^\alpha(\iota)\|d\iota\Big]$$
$$\leq \frac{1}{\mu}\frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left[\|z_c\| + \widetilde{J}_1P\widetilde{J}_2\|z_0\| + \widetilde{J}_1Pc\widetilde{J}_2\|z_1\| + \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^c (c-\iota)^{2a-1}L_{g,\alpha}(\iota)d\iota\right]$$

For such $\mu > 0$, we prove that

$$\alpha < \|(\mathcal{R}z^\alpha)(\varrho)\| \leq \|J^{-1}M_a(\varrho)Jz_0\| + \|J^{-1}W_a(\varrho)Jz_1\| + \int_0^\varrho (\varrho-\iota)^{a-1}\|J^{-1}Q_a(\varrho-\iota)g^\alpha(\iota)\|d\iota$$
$$+ \int_0^\varrho (\varrho-\iota)^{a-1}\|J^{-1}Q_a(\varrho-\iota)\mathscr{B}x^\alpha(\iota)\|d\iota$$
$$\leq \widetilde{J}_1P\widetilde{J}_2\|z_0\| + \widetilde{J}_1P\rho\widetilde{J}_2\|z_1\| + \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^\varrho (\varrho-\iota)^{2a-1}L_{g,\alpha}(\iota)d\iota$$
$$+ \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\int_0^\varrho (\varrho-\iota)^{2a-1}\frac{1}{\mu}\frac{\widetilde{J}_1PP_B}{\Gamma(2a)}\Big[\|z_c\| + \widetilde{J}_1P\widetilde{J}_2\|z_0\| + \widetilde{J}_1Pc\widetilde{J}_2\|z_1\|$$
$$+ \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^c (c-\iota)^{2a-1}L_{g,\alpha}(\iota)d\iota\Big]d\iota$$

$$\leq \widetilde{J}_1P\widetilde{J}_2\|z_0\| + \widetilde{J}_1P\varrho\widetilde{J}_2\|z_1\| + \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^\varrho (\varrho-\iota)^{2a-1}L_{g,\alpha}(\iota)d\iota$$
$$+ \frac{1}{\mu}\left(\frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\right)^2\frac{c^{2a}}{2a}\left[\|z_c\| + \widetilde{J}_1P\widetilde{J}_2\|z_0\| + \widetilde{J}_1Pc\widetilde{J}_2\|z_1\| + \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^c (c-\iota)^{2a-1}L_{g,\alpha}(\iota)d\iota\right]. \tag{3.4}$$

Dividing the Equation (3.4) by α, applying limit as $\alpha \to \infty$ to the above inequality and utilizing $\mathbf{H_4}$, we obtain

$$\frac{P\widetilde{J}_1}{\Gamma(2a)}\left[1 + \frac{1}{\mu}\left(\frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\right)^2\frac{c^{2a}}{2a}\right]\lambda \geq 1,$$

which contradicts our assumption. Hence $\mu > 0$, there exists $\alpha > 0$ such that $\mathcal{R}$ maps $\mathcal{Q}_\alpha \to \mathcal{Q}_\alpha$.

Step 3: $\mathcal{R}$ mapping bounded sets into equicontinuous sets of C.

For all $m \in \mathcal{R}(z)$ and $z \in \mathcal{Q}_\alpha$, there exists $g \in S_{E,z}$, we define

$$m(\varrho) = J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^\varrho (\varrho-\iota)^{a-1}J^{-1}Q_a(\varrho-\iota)g(\iota)d\iota$$
$$+ \int_0^\varrho (\varrho-\iota)^{a-1}J^{-1}Q_a(\varrho-\iota)\mathscr{B}x(\iota)d\iota.$$

Let $0 < \epsilon < \varrho < \varrho + h \leq c$.

Then,

$$\|m(\varrho+h) - m(\varrho)\| \leq \|J^{-1}[M_a(\varrho+h) - M_a(\varrho)]Jz_0\| + \|J^{-1}[W_a(\varrho+h) - W_a(\varrho)]Jz_1\|$$

$$
\begin{aligned}
&+\int_{\varrho}^{\varrho+h}(\varrho+h-\imath)^{a-1}\|J^{-1}Q_a(\varrho+h-\imath)\,g(\imath)\|d\imath\\
&+\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\imath)^{a-1}\|\left[J^{-1}Q_a(\varrho+h-\imath)-J^{-1}Q_a(\varrho-\imath)\right]g(\imath)\|d\imath\\
&+\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\imath)^{a-1}-(\varrho-\imath)^{a-1}\right]\|J^{-1}Q_a(\varrho-\imath)\,g(\imath)\|d\imath\\
&+\int_{0}^{\varrho-\epsilon}(\varrho+h-\imath)^{a-1}\|\left[J^{-1}Q_a(\varrho+h-\imath)-J^{-1}Q_a(\varrho-\imath)\right]g(\imath)\|d\imath\\
&+\int_{0}^{\varrho-\epsilon}\left[(\varrho+h-\imath)^{a-1}-(\varrho-\imath)^{a-1}\right]\|J^{-1}Q_a(\varrho-\imath)\,g(\imath)\|d\imath\\
&+\int_{\varrho}^{\varrho+h}(\varrho+h-\imath)^{a-1}\|J^{-1}Q_a(\varrho+h-\imath)\,\mathscr{B}x(\imath)\|d\imath\\
&+\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\imath)^{a-1}\|\left[J^{-1}Q_a(\varrho+h-\imath)-J^{-1}Q_a(\varrho-\imath)\right]\mathscr{B}x(\imath)\|d\imath\\
&+\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\imath)^{a-1}-(\varrho-\imath)^{a-1}\right]\|J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}x(\imath)\|d\imath\\
&+\int_{0}^{\varrho-\epsilon}(\varrho+h-\imath)^{a-1}\|\left[J^{-1}Q_a(\varrho+h-\imath)-J^{-1}Q_a(\varrho-\imath)\right]\mathscr{B}x(\imath)\|d\imath\\
&+\int_{0}^{\varrho-\epsilon}\left[(\varrho+h-\imath)^{a-1}-(\varrho-\imath)^{a-1}\right]\|J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}x(\imath)\|d\imath\\
=&\sum_{i=1}^{12}\mathcal{O}_i.
\end{aligned}
$$

Let $b=\frac{2a-1}{1-v}\in(-1,0)$. By applying $\mathbf{H_1}$, $\mathbf{H_3}$ and Lemma 2.10 for $\mathcal{O}_3,\mathcal{O}_4,\mathcal{O}_5,\mathcal{O}_6$ and $\mathcal{O}_7$, we have

$$
\begin{aligned}
\mathcal{O}_3 &\le \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho}^{\varrho+h}(\varrho+h-\imath)^{2a-1}\|g(\imath)\|d\imath\\
&\le \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho}^{\varrho+h}(\varrho+h-\imath)^{2a-1}L_{g,\alpha}(\imath)\,d\imath\\
&\le \frac{P\widetilde{J}_1}{\Gamma(2a)}\left(\int_{\varrho}^{\varrho+h}(\varrho+h-\imath)^{\frac{2a-1}{1-v}}d\imath\right)^{1-v}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)}\\
&\le \frac{P\widetilde{J}_1}{\Gamma(2a)}\left[\frac{h^{(b+1)(1-v)}}{(b+1)^{1-v}}\right]\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_4 &\le \frac{2P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\imath)^{2a-1}\|g(\imath)\|d\imath\\
&\le \frac{2P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\imath)^{2a-1}L_{g,\alpha}(\imath)\,d\imath\\
&\le \frac{2P\widetilde{J}_1}{\Gamma(2a)}\left(\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\imath)^{\frac{2a-1}{1-v}}d\imath\right)^{1-v}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)}
\end{aligned}
$$

$$\leq \frac{2P\widetilde{J}_1}{\Gamma(2a)}\left[\frac{\epsilon^{(b+1)(1-v)}}{(b+1)^{1-v}}\right]\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)},$$

$$\begin{aligned}\mathcal{O}_5 &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|g(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\left(\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]^{\frac{1}{1-v}}d\iota\right)^{1-v}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)}\\ &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\left[\frac{2h^{(b+1)(1-v)}}{(b+1)^{1-v}}\right]\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)},\end{aligned}$$

$$\begin{aligned}\mathcal{O}_6 &\leq \sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{2a-1}\|g(\iota)\|d\iota\\ &\leq \sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{2a-1}L_{g,\alpha}(\iota)\,d\iota\\ &\leq \sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\left(\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{\frac{2a-1}{1-v}}d\iota\right)^{1-v}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)}\\ &\leq \sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\frac{\left[(\varrho+h)^{b+1}-(h+\epsilon)^{b+1}\right]^{1-v}}{(b+1)^{1-v}}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)},\end{aligned}$$

$$\begin{aligned}\mathcal{O}_7 &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|g(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]L_{g,\alpha}(\iota)\,d\iota\\ &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\left(\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]^{\frac{1}{1-v}}d\iota\right)^{1-v}\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)}\\ &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\left[\frac{2h^{(b+1)(1-v)}}{(b+1)^{1-v}}\right]\|L_{g,\alpha}\|_{L^{\frac{1}{v}}\left(V,\mathbb{R}^+\right)},\end{aligned}$$

For $\mathcal{O}_8, \mathcal{O}_9, \mathcal{O}_{10}, \mathcal{O}_{11}$ and $\mathcal{O}_{12}$, using the $\mathbf{H_1}$, $\mathbf{H_3}$ and Lemma 2.10

$$\begin{aligned}\mathcal{O}_8 &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho}^{\varrho+h}(\varrho+h-\iota)^{2a-1}\|\mathscr{B}x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\int_{\varrho}^{\varrho+h}(\varrho+h-\iota)^{2a-1}\|x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left(\int_{\varrho}^{\varrho+h}(\varrho+h-\iota)^{\frac{2a-1}{1-v}}d\iota\right)^{1-v}\|x\|\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left[\frac{h^{(b+1)(1-v)}}{(b+1)^{1-v}}\right]\|x\|,\end{aligned}$$

$$\begin{aligned}\mathcal{O}_9 &\leq \frac{2P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\iota)^{2a-1}\|\mathscr{B}x(\iota)\|d\iota\\ &\leq \frac{2P\widetilde{J}_1P_B}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\iota)^{2a-1}\|x(\iota)\|d\iota\end{aligned}$$

$$\leq \frac{2P\widetilde{J}_1P_B}{\Gamma(2a)}\left(\int_{\varrho-\epsilon}^{\varrho}(\varrho+h-\iota)^{\frac{2a-1}{1-\nu}}d\iota\right)^{1-\nu}\|x\|$$
$$\leq \frac{2P\widetilde{J}_1P_B}{\Gamma(2a)}\left[\frac{\epsilon^{(b+1)(1-\nu)}}{(b+1)^{1-\nu}}\right]\|x\|,$$

$$\begin{aligned}\mathcal{O}_{10} &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|\mathscr{B}x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left(\int_{\varrho-\epsilon}^{\varrho}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]^{\frac{1}{1-\nu}}d\iota\right)^{1-\nu}\|x\|\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left[\frac{2h^{(b+1)(1-\nu)}}{(b+1)^{1-\nu}}\right]\|x\|,\end{aligned}$$

$$\begin{aligned}\mathcal{O}_{11} &\leq \sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{2a-1}\|\mathscr{B}x(\iota)\|d\iota\\ &\leq P_B\sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{2a-1}\|x(\iota)\|d\iota\\ &\leq P_B\sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\left(\int_0^{\varrho-\epsilon}(\varrho+h-\iota)^{\frac{2a-1}{1-\nu}}d\iota\right)^{1-\nu}\|x\|\\ &\leq P_B\sup_{\iota\in[0,\varrho-\epsilon]}\|J^{-1}Q_a(\varrho+h-\iota)-J^{-1}Q_a(\varrho-\iota)\|\frac{\left[(\varrho+h)^{b+1}-(h+\epsilon)^{b+1}\right]^{1-\nu}}{(b+1)^{1-\nu}}\|x\|,\end{aligned}$$

$$\begin{aligned}\mathcal{O}_{12} &\leq \frac{P\widetilde{J}_1}{\Gamma(2a)}\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|\mathscr{B}x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]\|x(\iota)\|d\iota\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left(\int_0^{\varrho-\epsilon}\left[(\varrho+h-\iota)^{2a-1}-(\varrho-\iota)^{2a-1}\right]^{\frac{1}{1-\nu}}d\iota\right)^{1-\nu}\|x\|\\ &\leq \frac{P\widetilde{J}_1P_B}{\Gamma(2a)}\left[\frac{2h^{(b+1)(1-\nu)}}{(b+1)^{1-\nu}}\right]\|x\|.\end{aligned}$$

It is easy to verify $\mathcal{O}_3$-$\mathcal{O}_5, \mathcal{O}_7$-$\mathcal{O}_{10}, \mathcal{O}_{12} \to 0$ as $h \to 0$. Additionally, by referring the compactness of $T(\varrho), \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_6, \mathcal{O}_{11}$ tends to zero. Therefore

$$\|m(\varrho+h)-m(\varrho)\| \to 0$$

when $h \to 0$ for all $z \in \mathcal{Q}_\alpha$, which implies $\mathcal{R}(\mathcal{Q}_\alpha) \subset \mathcal{C}$ is equicontinuous.

Step 4: For $\alpha > 0$, fix $W_\alpha = \{z \in \mathscr{Y} : |z| \leq \alpha\}$. Clearly, W_α a bounded subset in $\mathscr{Y}$. To prove for all $\alpha > 0$ and $\rho > 0$,

$$\mathscr{U}(\varrho) = \left\{\int_0^\infty a\rho J^{-1}S_a(\rho)S(\varrho^a\rho)z\,d\rho,\ \ z \in W_\alpha\right\},$$

are relatively compact in $\mathscr{Y}$.

Assume that $\rho > 0$ be determined. For every $\delta > 0$, $0 < \varepsilon \leq \varrho$, determine a subset in $\mathscr{Y}$ by

$$\mathscr{U}_{\varepsilon,\delta}(\varrho) = \left\{ \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} \int_\delta^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho - \varepsilon^a\delta) z d\rho, \quad z \in W_\alpha \right\}.$$

Clearly for every $\varrho > 0$, $\mathscr{U}_{\varepsilon,\delta}(\varrho)$ is clearly defined. Indeed, referring uniform convergence of Mainardi's Wright-type function $\rho \in (\delta, \infty)$ and uniform boundedness of cosine family and, we get for every $z \in W_\alpha$,

$$\left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} \int_\delta^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho - \varepsilon^a\delta) z d\rho \right| \leq \widetilde{J}_1 P^2 \mid z \mid \int_\delta^\infty a\rho S_a(\rho)(\varrho^a\rho + \varepsilon^a\delta) d\rho$$
$$\leq 2\widetilde{J}_1 P^2 \mid z \mid \varrho^a \int_\delta^\infty a\rho^2 S_a(\rho) d\rho \leq \frac{2\widetilde{J}_1 P^2}{\Gamma(2a)} \mid z \mid \varrho^a.$$

Therefore, $\mathscr{U}_{\varepsilon,\delta}(\varrho)$ is relatively compact because $S(\varepsilon^a\delta)$ is compact for $\varepsilon^a\delta > 0$.
Additionally,

$$\left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} \int_\delta^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho - \varepsilon^a\delta) z d\rho - \int_0^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho) z d\rho \right|$$
$$\leq \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} \int_\delta^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho - \varepsilon^a\delta) z d\rho - \int_0^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho) z d\rho \right|$$
$$+ \left| \int_\delta^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho) z d\rho - \int_0^\infty a\rho J^{-1} S_a(\rho) S(\varrho^a\rho) z d\rho \right|$$
$$\leq \int_\delta^\infty a\rho \widetilde{J}_1 S_a(\rho) \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z \right| d\rho + \int_0^\delta a\rho \widetilde{J}_1 S_a(\rho) \mid S(\varrho^a\rho) z \mid d\rho$$
$$:= l_1 + l_2.$$

Since

$$a\rho\widetilde{J}_1 S_a(\rho) \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z \right| \leq 2\widetilde{J}_1 P^2 \varrho^a a\rho^2 S_a(\rho) \mid z \mid,$$

and

$$\int_0^\infty a\rho^2 \widetilde{J}_1 S_a(\rho) d\rho = \frac{2\widetilde{J}_1}{\Gamma(1+2a)},$$

we can see that

$$\int_0^\infty a\rho\widetilde{J}_1 S_a(\rho) \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z \right| d\rho,$$

is uniformly convergence. Since from the strongly continuous of sine family $\{S(\varrho)\}_{\varrho > 0}$, where $\rho \in (\delta, \infty)$, by referring Lemma 2.12, we have

$$\left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z \right|$$
$$\leq \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho - \varepsilon^a\delta) z \right| + |S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z| \to 0,$$

when $\delta \to 0$. Therefore, we have

$$l_1 \leq \int_0^\infty a\rho\widetilde{J}_1 S_a(\rho) \left| \frac{S(\varepsilon^a\delta)}{\varepsilon^a\delta} S(\varrho^a\rho - \varepsilon^a\delta) z - S(\varrho^a\rho) z \right| d\rho \to 0, \quad \text{as} \quad \delta \to 0.$$

However, by $\int_0^\delta a\rho^2 \widetilde{J}_1 S_a(\rho) d\rho \to 0$ when $\delta \to 0$, we have

$$l_2 \leq \widetilde{J}_1 P \mid z \mid \rho^a \int_0^\delta a\rho^2 S_a(\rho) d\rho \to 0, \quad \text{when} \quad \delta \to 0.$$

Thus, there are relatively compact sets arbitrarily close to $\mathscr{U}(\varrho)$, for all $\varrho > 0$. Thus $\mathscr{U}(\varrho)$ is relatively compact in $\mathscr{Y}$, for all $\varrho > 0$.

Step 5: $\mathcal{R}$ has a closed graph.

Assume that $z^n \to z^*$ when $n \to \infty$, and $m^n \to m^*$ when $n \to \infty$. We shall prove that $m^* \in \mathcal{R}(z^*)$. Since $m^n \in \mathcal{R}(z^n)$, there exists $g^n \in S_{E,z^n}$ such that

$$\begin{aligned} m^n(\varrho) &= J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,g^n(\imath)\,d\imath \\ &+ \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho)\,R\left(\mu,\Gamma_0^c\right)\left[z_c - J^{-1}M_a(c)Jz_0\right. \\ &\left. - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^n(\varphi)\,d\varphi\right](\imath)\,d\imath. \end{aligned}$$

We need to show that there exists $g^* \in S_{E,z^*}$ such that for all $\varrho \in V$,

$$\begin{aligned} m^*(\varrho) &= J^{-1}M_a(\varrho)Jz_0 + J^{-1}W_a(\varrho)Jz_1 + \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,g^*(\imath)\,d\imath \\ &+ \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho)\,R\left(\mu,\Gamma_0^c\right) \\ &\times \left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^*(\varphi)\,d\varphi\right](\imath)\,d\imath, \end{aligned}$$

clearly,

$$\begin{aligned} &\Bigg\|\Bigg(m^n(\varrho) - J^{-1}M_a(\varrho)Jz_0 - J^{-1}W_a(\varrho)Jz_1 - \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho) \\ &(\times)R\big(\mu,\Gamma_0^c\big)\left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^n(\varphi)\,d\varphi\right](\imath)\,d\imath\Bigg) \\ &- \Bigg(m^*(\varrho) - J^{-1}M_a(\varrho)Jz_0 - J^{-1}W_a(\varrho)Jz_1 - \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho) \\ &(\times)R\big(\mu,\Gamma_0^c\big)\left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^*(\varphi)\,d\varphi\right](\imath)\,d\imath\Bigg)\Bigg\| \end{aligned}$$

$\to 0$ as $n \to \infty$. Assume that $\mathcal{T} : L^1(V, \mathscr{Y}) \to C$,

$$\begin{aligned} (\mathcal{T}g)(\varrho) &= \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\left[g^n(\imath) + \mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho)R\big(\mu,\Gamma_0^c\big)\right. \\ &\left.\times\left(\int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^*(\varphi)\,d\varphi\right)(\imath)\right]d\imath \end{aligned}$$

clearly, from Lemma 2.14, that $\mathcal{T} \circ S_{E,z}$ is a closed graph operator. Additionally, by referring $\mathcal{T}$, we have

$$\begin{aligned} &\Big(m^n(\varrho) - J^{-1}M_a(\varrho)Jz_0 - J^{-1}W_a(\varrho)Jz_1 - \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho)\,R\left(\mu,\Gamma_0^c\right) \\ &\times\left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)\,g^n(\varphi)\,d\varphi\right](\imath)\,d\imath\Big) \\ &\in \mathcal{T}\left(S_{E,z^n}\right) \end{aligned}$$

Because $g^n \to g^*$, and by referring Lemma 2.14, we have

$$\Big(m^*(\varrho) - J^{-1}M_a(\varrho)Jz_0 - J^{-1}W_a(\varrho)Jz_1 - \int_0^{\varrho} (\varrho-\imath)^{a-1}J^{-1}Q_a(\varrho-\imath)\,\mathscr{B}\mathscr{B}^*J^{-1}Q_a^*(c-\varrho)\,R\left(\mu,\Gamma_0^c\right)$$

$$\times \left[z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\varphi)^{a-1}J^{-1}Q_a(c-\varphi)g^*(\varphi)\,d\varphi \right](\iota)\,d\iota\Big)$$
$$\in \mathcal{T}\left(S_{E,z^*}\right)$$

Hence, $\mathcal{R}$ is closed graph.

Hence, by referring **Step 1–5**, and applying Arzela-Ascoli theorem, $\mathcal{R}$ is a completely continuous multivalued mapping with compact value and thus $\mathcal{R}$ is u.s.c. Therefore $\mathcal{R}$ has a fixed point $z(\cdot)$ on $\mathcal{Q}_\alpha$, and in view of Lemma 2.15, $z(\cdot)$ is the mild solution of (1.1)–(1.2). □

Definition 3.2 The system (1.1)–(1.2) is called approximately controllable on V if $\overline{R(c,z_0)} = \mathscr{Y}$, $R(c,z_0) = \left\{ z_c(z_0;x) : x(\cdot) \in L^2(V,\mathscr{U}) \right\}$ is a solution of (1.1)–(1.2). □

Theorem 3.3 *Assume that* $\mathbf{H_1} - \mathbf{H_5}$ *are satisfied. Further, there exists*

$$\mathscr{I} \in L^1\left(V,\left[0,+\infty\right)\right) \text{ such that } \sup_{z\in\mathcal{Y}} \| E(\varrho,z) \| \leq \mathscr{I}(\varrho)$$

for a.e. $\varrho \in V$, then (1.1)–(1.2) is approximately controllable.

Proof. Consider $\hat{z}^\mu(\cdot)$ a fixed point of $\mathcal{R}$ in $\mathcal{Q}_\alpha$. Referring 3.1, any fixed point of $\mathcal{R}$ is a solution of (1.1)–(1.2) with

$$\hat{x}^\mu(\rho) = \mathscr{B}^* J^{-1} Q_a^*(c-\varrho) R\left(\mu,\Gamma_0^c\right) q\left(\hat{z}^\mu\right)$$

and satisfies the following inequality

$$\hat{z}^\mu(c) = z_c + \mu R\left(\mu,\Gamma_0^c\right) q\left(\hat{z}^\mu\right). \tag{3.5}$$

Further from E and Dunford-Pettis Theorem, we conclude that $\{g^\mu(\iota)\}$ is weakly compact in $L^1(V,\mathscr{Y})$, so there exists a subsequence, $\{g^\mu(\iota)\}$, that converges weakly to $g(\iota)$ in $L^1(V,\mathscr{Y})$. Determine

$$\mathcal{V} = z_c - J^{-1}M_a(c)Jz_0 - J^{-1}W_a(c)Jz_1 - \int_0^c (c-\iota)^{a-1}J^{-1}Q_a(c-\iota)g(\iota)\,d\iota.$$

Now, we have

$$\begin{aligned} \|q\left(\hat{z}^\mu\right) - \mathcal{V}\| &= \| \int_o^c (c-\iota)^{a-1}J^{-1}Q_a(c-\iota)\left[g\left(\iota,\hat{z}^\mu(\iota)\right) - g(\iota)\right] d\iota \| \\ &\leq \sup_{\varrho\in V} \| \int_o^\varrho (\varrho-\iota)^{a-1}J^{-1}Q_a(\varrho-\iota)\left[g\left(\iota,\hat{z}^\mu(\iota)\right) - g(\iota)\right] d\iota \|. \end{aligned} \tag{3.6}$$

By Ascoli-Arzela theorem, we can prove

$$\mathcal{L}(\cdot) \to \int_0^\cdot (\cdot-\iota)^{a-1}J^{-1}Q_a(\cdot-\iota)\mathscr{L}(\iota)\,d\iota : L^1(V,\mathscr{Y}) \to C(V,\mathscr{Y})$$

is compact. Hence, we have

$$\|q\left(\hat{z}^\mu\right) - \mathcal{V}\| \to 0 \text{ when } \mu \to 0^+.$$

Furthermore, in view of Equation (3.5), we have

$$\begin{aligned} \|\hat{z}^\mu(c) - z_c\| &\leq \|\mu R\left(\mu,\Gamma_0^c\right)(\mathcal{V})\| + \|\mu R\left(\mu,\Gamma_0^c\right)\| \|q\left(\hat{z}^\mu\right) - \mathcal{V}\| \\ &\leq \|\mu R\left(\mu,\Gamma_0^c\right)(\mathcal{V})\| + \|q\left(\hat{z}^\mu\right) - \mathcal{V}\|. \end{aligned}$$

It follows from $\mathbf{H_0}$ and (3.6) that

$$\|\hat{z}^\mu(c) - z_c\| \to 0 \text{ as } \mu \to 0^+.$$

Therefore the system (1.1)–(1.2) is approximately controllable on V. □

4 | NONLOCAL CONDITIONS

Differential systems with nonlocal conditions have become a very lively field of recent research. Their investigation is driven by hypothetical enthusiasm as well as by the way that these kinds of issues normally happen when demonstrating practical applications. For instance, few events in designing, material science, and life sciences can be portrayed by methods for the differential system subject to nonlocal boundary conditions, and one can refer [6, 9, 11, 14, 16, 18, 25, 32, 37–41]. Consider the nonlocal fractional Sobolev type Volterra-Fredholm integro-differential inclusions of order $r \in (1, 2)$ of the type

$$ {}^{C}D^{r}_{\varrho}\left(Jz\left(\varrho\right)\right) \in Az\left(\varrho\right) + \mathscr{B}x\left(\varrho\right) + E\left(\varrho, z\left(\varrho\right), \int_0^{\varrho} e(\varrho, s, z(s))ds, \int_0^{c} h(\varrho, s, z(s))\, ds\right), \quad \varrho \in V, \tag{4.1} $$

$$ z\left(0\right) + l\left(z\right) = z_0, \;\; z'\left(0\right) = z_1 \in \mathscr{Y}, \tag{4.2} $$

where the function $l : C\left([0, c], \mathscr{Y}\right) \to \mathscr{Y}$ satisfies the following hypothesis:

$\mathbf{H_6}$ The function $l : C\left([0, c], \mathscr{Y}\right) \to \mathscr{Y}$ is compact and continuous, and there exists constants L_1, L_2 such that $\|l(z)\| \leq L_1 \|z\|_C + L_2$, where $z \in C\left(V, \mathscr{Y}\right)$.

The mild solution of the nonlocal fractional evolution system (4.1)–(4.2) defined as follows:

Definition 4.1 A function $z \in C = C\left(V, \mathscr{Y}\right)$ is called a solution of (4.1)–(4.2) if $z(0) + l(z) = z_0$, $z'\left(0\right) = z_1, x\left(\cdot\right) \in L^2\left(V, \mathscr{U}\right)$ and there exists $g \in L^1\left(V, \mathscr{Y}\right)$ such that $g\left(\varrho\right) \in E\left(\varrho, z\left(\rho\right), \int_0^{\varrho} e\left(\varrho, s, z\left(s\right)\right) ds, \int_0^{c} h\left(\varrho, s, z\left(s\right)\right) ds\right)$ on a.e. $\varrho \in V$ and

$$ \begin{aligned} z\left(\varrho\right) = {} & J^{-1}M_a\left(\varrho\right) J\left[z_0 - l\left(z\right)\right] + J^{-1}W_a\left(\varrho\right) Jz_1 + \int_0^{\varrho} \left(\varrho - \imath\right)^{a-1} J^{-1} Q_a\left(\varrho - \imath\right) \mathscr{B}x\left(\imath\right) d\imath \\ & + \int_0^{\rho} \left(\varrho - \imath\right)^{a-1} J^{-1} Q_a\left(\varrho - \imath\right) g\left(\imath\right) d\imath, \end{aligned} $$

is satisfied.

Theorem 4.2 *If* $\mathbf{H_1}$–$\mathbf{H_6}$ *are satisfied, then the system* (4.1)–(4.2) *is approximately controllable on* V.

5 | EXAMPLE

Let us consider $\chi \subset \mathbb{R}^N$ be an open C^2 bounded domain. Assume that $\mathscr{Y} = \mathscr{U} = L^2\left(\chi\right)$. Let us assume the fractional integro-differential system

$$ \begin{aligned} & \partial^{r}_{\rho}\left(u\left(\varrho, v\right) - \Delta u\left(\varrho, v\right)\right) \in \Delta u\left(\varrho, v\right) + \hbar\left(\varrho, v\right) \\ & \quad + P\left(\varrho, u\left(\varrho, z\right), \int_0^{\varrho} e\left(\varrho, s, u\left(\varrho, z\right)\right) ds, \int_0^{c} h\left(\varrho, s, u\left(\varrho, z\right)\right) ds\right), \quad \varrho \in [0, 1], v \in \chi, \end{aligned} \tag{5.1} $$

$$ u\left(\varrho, v\right) = 0, \quad \varrho \in [0, 1], \quad v \in \partial\chi, \tag{5.2} $$

$$ u\left(0, v\right) = u_0\left(v\right), \;\; u'\left(0, v\right) = u_1\left(v\right), v \in \chi. \tag{5.3} $$

In the above, ∂_ϱ^r is the partial derivative in Caputo sense of fractional order $r \in (1,2)$. Let us consider $V = [0, 1]$. Assume that A be the Laplace operator with Dirichlet boundary conditions given as $Au = \Delta$ and assume $A : D(A) \subset \mathscr{Y} \to \mathscr{Y}$, $J : D(J) \subset \mathscr{Y} \to \mathscr{Y}$ be the operators defined by $Au = \Delta$, and $Ju = u - \Delta$ where each domain $D(A)$ and $D(J)$ is presented by

$$D(A) = D(J) = \left\{ g \in H_0^1(\chi),\ \ Ag \in L^2(\chi) \right\}.$$

It is clear that $D(A) = H_0^1(\chi) \cap H^2(\chi)$. Obviously A gives a uniformly bounded C_0 cosine family $C(\varrho)$ for $\varrho \geq 0$, refer [42]. Indeed, let $\iota_n = n^2\pi^2$ and $\psi_n(v) = \sqrt{\frac{2}{\pi}} \sin(n\pi v)$, for all $n \in \mathbb{N}$ are the orthonormal of vectors of A.

Assume $\{-\iota_n, \psi_n\}_{n=1}^{\infty}$ is the eigensystem of the operator A, then $0 < \iota_1 \leq \iota_2 \leq \cdots$, $\iota_n \to \infty$ as $n \to \infty$, and $\{\psi_n\}_{n=1}^{\infty}$ create the orthonormal basis of $\mathscr{Y}$. Now

$$Au = \sum_{n=1}^{\infty} \iota_n \langle u, \psi_n \rangle \psi_n, \quad u \in D(A),$$

$$Ju = \sum_{n=1}^{\infty} (1 + \iota_n) \langle u, \psi_n \rangle \psi_n, \quad u \in D(J).$$

Additionally, for $z \in \mathscr{Y}$, we have

$$J^{-1}u = \sum_{n=1}^{\infty} \frac{1}{(1 + \iota_n)} \langle u, \psi_n \rangle \psi_n,$$

$$AJ^{-1}u = \sum_{n=1}^{\infty} \frac{\iota_n}{(1 + \iota_n)} \langle u, \psi_n \rangle \psi_n,$$

where $\langle \cdot, \cdot \rangle$ stands for inner product in $\mathscr{Y}$. Accordingly, we now define the cosine family by

$$C(\varrho)u = \sum_{n=1}^{\infty} \cos\left(\sqrt{\iota_n}\varrho\right) \langle u, \psi_n \rangle \psi_n, \quad u \in \mathscr{Y},$$

which is connected with sine family $S(\varrho)$ compact for $(\varrho > 0)$ as follows

$$S(\varrho)y = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\iota_n}} \sin\left(\sqrt{\iota_n}\varrho\right) \langle u, \psi_n \rangle \psi_n, \quad u \in \mathscr{Y}.$$

It is not difficult to verify $\|C(\varrho)\|_{L_c(\mathscr{Y})} \leq 1$, for all $\varrho \in \mathbb{R}$.

Let us assume $u(\varrho) = u(\varrho, \cdot)$, that is, $u(\varrho)(v) = u(\varrho, v)$, $\varrho \in [0, 1]$, and $x(\varrho) = \hbar(\varrho, \cdot)$, consider $\hbar : [0, 1] \times \chi \to \chi$ is continuous. Determine $\mathscr{B} : \mathscr{U} \to \mathscr{Y}$ by $\mathscr{B}x(\varrho)(v) = \hbar(\varrho, v)$. We now define

$$\begin{aligned} &E\left(\varrho, u(\varrho), \int_0^{\varrho} e(\varrho, s, u(s))\,ds, \int_0^{c} h(\varrho, s, u(s))\,ds\right)(z) \\ &\quad = P\left(\varrho, u(\varrho, z), \int_0^{\varrho} e(\varrho, s, u(\varrho, z))\,ds, \int_0^{c} h(\varrho, s, u(\varrho, z))\,ds\right). \end{aligned}$$

Therefore, entire needs of Theorem 3.1 are satisfied, hence (5.1)–(5.3) is approximately controllable on V.

6 | CONCLUSION

In our article, we primarily concentrated on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential inclusions of order $r \in (1, 2)$. By applying the results and ideas belongs to the cosine function of operators, fractional calculus and fixed point approach, the main results are established. Initially, we established the approximate controllability of the considered fractional system, then continued to examine the system with the concept of nonlocal conditions. Finally, we presented an example to demonstrate the theory.

CONFLICT OF INTEREST

None declared.

ORCID

V. Vijayakumar https://orcid.org/0000-0001-5976-5794
Chokkalingam Ravichandran https://orcid.org/0000-0003-0214-1280
Kottakkaran Sooppy Nisar https://orcid.org/0000-0001-5769-4320
Kishor D. Kucche https://orcid.org/0000-0002-7141-330X

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
[2] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of fractional dynamic systems*, Cambridge scientific, Cambridge, UK, 2009.
[3] Y. Zhou, *Basic theory of fractional differential equations*, World Scientific, Singapore, 2014.
[4] Y. Zhou, *Fractional evolution equations and inclusions: Analysis and control*, Elsevier, New York, 2015.
[5] M. A. Alqudah, C. Ravichandran, T. Abdeljawad, and N. Valliammal, *New results on Caputo fractional-order neutral differential inclusions without compactness*, Adv. Differ. Eq. 2019(1) (2019), 1–14.
[6] K. Kavitha, V. Vijayakumar, and R. Udhayakumar, *Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness*, Chaos Solitons Fractals 139 (2020), 110035.
[7] K. Kavitha, V. Vijayakumar, R. Udhayakumar, and K. S. Nisar, *Results on the existence of Hilfer fractional neutral evolution equations with infinite delay via measures of noncompactness*, Math. Methods Appl. Sci. 44 (2020), 1–18. https://doi.org/10.1002/mma.6843.
[8] J. A. Machado, C. Ravichandran, and M. Rivero, *Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions*, Fixed Point Theory Appl. 2013(66) (2013), 1–17.
[9] N. I. Mahmudov, R. Murugesu, C. Ravichandran, and V. Vijayakumar, *Approximate controllability results for fractional semilinear integro-differential inclusions in Hilbert spaces*, Results Math. 71(1) (2017), 45–61.
[10] M. Mohan Raja, V. Vijayakumar, and R. Udhayakumar, *Results on the existence and controllability of fractional integro-differential system of order 1 < r < 2 via measure of noncompactness*, Chaos Solitons Fractals 139 (2020), 1–11 Article ID 110299.
[11] G. M. Mophou and G. M. N'Guerekata, *Existence of mild solution for some fractional diffferential equations with nonlocal conditions*, Semigroup Forum 79(2) (2009), 322–335.
[12] C. Ravichandran, N. Valliammal, and J. J. Nieto, *New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces*, J. Franklin Inst. 356(3) (2019), 1535–1565.
[13] R. Sakthivel, R. Ganesh, and S. M. Anthoni, *Approximate controllability of fractional nonlinear differential inclusions*, Appl. Math. Comput. 225 (2013), 708–717.
[14] V. Vijayakumar, *Approximate controllability results for non-densely defined fractional neutral differential inclusions with Hille-Yosida operators*, Int. J. Control. 92(9) (2019), 2210–2222.
[15] V. Vijayakumar and R. Udhayakumar, *A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay*, Numer. Methods Partial Differ. Eq. 37 (2020), 1–17. https://doi.org/10.1002/num.22550.
[16] V. Vijayakumar and R. Udhayakumar, *Results on approximate controllability for non-densely defined Hilfer fractional differential system with infinite delay*, Chaos, Solitons Fractals 139 (2020), 110019.
[17] W. K. Williams, V. Vijayakumar, R. Udhayakumar, and K. S. Nisar, *A new study on existence and uniqueness of nonlocal fractional delay differential systems of order 1 < r < 2 in Banach spaces*, Numer. Methods Partial Differ. Eq. (2020), 1–13. https://doi.org/10.1002/num.22560.
[18] Y. Zhou, V. Vijayakumar, C. Ravichandran, and R. Murugesu, *Controllability results for fractional order neutral functional differential inclusions with infinite delay*, Fixed Point Theory 18(2) (2017), 773–798.
[19] Z. Zufeng and B. Liu, *Controllability results for fractional functional differential equations with nondense domain*, Numer. Funct. Anal. Optim. 35 (2014), 443–460.

[20] Y. K. Chang and D. N. Chalishajar, *Controllability of mixed Volterra-Fredholm-type integro-differential inclusions in Banach spaces*, J. Franklin Inst. 345 (2008), 499–507.

[21] D. N. Chalishajar, *Controllability of mixed Volterra-Fredholm-type integro-differential systems in Banach space*, J. Franklin Inst. 344 (2007), 12–21.

[22] N. I. Mahmudov and A. Denker, *On controllability of linear stochastic systems*, Int. J. Control. 73 (2000), 144–151.

[23] V. Vijayakumar, *Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces*, IMA J. Math. Control. Inf. 35(1) (2018), 297–314.

[24] V. Vijayakumar, *Approximate controllability results for impulsive neutral differential inclusions of Sobolev-type with infinite delay*, Int. J. Control. 91(10) (2018), 2366–2386.

[25] J. W. He, Y. Liang, B. Ahmad, and Y. Zhou, *Nonlocal fractional evolution inclusions of order $\alpha \in (1, 2)$*, Mathematics 209(7) (2019), 1–17.

[26] K. Li, J. Peng, and J. Gao, *Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces*, Rep. Math. Phys. 71 (2013), 33–43.

[27] Y. Li, *Regularity of mild solutions for fractional abstract Cauchy problem with order $\alpha \in (1, 2)$*, Z. Angew. Math. Phys. 66 (2015), 3283–3298.

[28] Y. Li, H. Sun, and Z. Feng, *Fractional abstract Cauchy problem with order $\alpha \in (1, 2)$*, Dyn. Partial Differ. Eq. 13 (2016), 155–177.

[29] M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, and Y. Zhou, *A new approach on the approximate controllability of fractional differential evolution equations of order $1 < \mathrm{r} < 2$ in Hilbert spaces*, Chaos Solitons Fractals 141 (2020), 1102310.

[30] M. Mohan Raja, V. Vijayakumar, and R. Udhayakumar, *A new approach on approximate controllability of fractional evolution inclusions of order $1 < \mathrm{r} < 2$ with infinite delay*, Chaos Solitons Fractals 141 (2020), 110343.

[31] X. B. Shu and F. Xu, *Upper and lower solution method for fractional evolution equations with order $1 < \alpha < 2$*, J. Korean Math. Soc. 51(6) (2014), 1123–1139.

[32] X. B. Shu and Q. Wang, *The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$*, Comput. Math. Appl. 64 (2012), 2100–2110.

[33] Y. Zhou and J. W. He, *New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$*, Evol. Eq. Control Theory 0 (2019), 1–19. https://doi.org/10.3934/eect.2020077.

[34] J. H. Lightbourne and S. Rankin, *A partial functional differential equation of Sobolev type*, J. Math. Anal. Appl. 93 (1983), 328–337.

[35] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Hungar. 32 (1978), 75–96.

[36] J. Wang and Y. Zhou, *Existence and controllability results for fractional semilinear differential inclusions*, Nonlinear Anal.: RWA 12 (2011), 3642–3653.

[37] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. 162 (1991), 494–505.

[38] L. Byszewski and H. Akca, *On a mild solution of a semilinear functional-differential evolution nonlocal problem*, J. Appl. Math. Stoch. Anal. 10(3) (1997), 265–271.

[39] A. Debbouche and D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl. 62 (2011), 1442–1450.

[40] G. M. Mophou and G. M. N'Guerekata, *On integral solutions of some nonlocal fractional differential equations with nondense domain*, Nonlinear Anal. 71(10) (2009), 4668–4675.

[41] G. M. N'Guerekata, *A Cauchy problem for some fractional abstract differential equation with nonlocal conditions*, Nonlinear Anal.: TMA 70(5) (2009), 1873–1876.

[42] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, 2nd ed., Birkhauser Verlag, Basel AG, 2011.

[43] V. Vijayakumar, and R. Udhayakumar, *Results on approximate controllability for non-densely defined Hilfer fractional differential system with infinite delay*, Chaos, Solitons & Fractals. 139 (2020), 110019. http://dx.doi.org/10.1016/j.chaos.2020.110019.

[44] V. Vijayakumar, and R. Udhayakumar, *A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay*, Numer. Methods Partial Differ. Eq. 37(1), (2021), 750–766. http://dx.doi.org/10.1002/num.22550.

[45] W. W. Kavitha, V. Vijayakumar, R. Udhayakumar, and K. S. Nisar, *A new study on existence and uniqueness of nonlocal fractional delay differential systems of order $1 < \mathrm{r} < 2$ in Banach spaces*, Numer. Methods Partial Differ. Eq. (2020), 1–13. http://dx.doi.org/10.1002/num.22560.

[46] V. Vijayakumar, R. Udhayakumar, Y. Zhou, and N. Sakthivel, *Approximate controllability results for Sobolev-type delay differential system of fractional order without uniqueness*, Numer. Methods Partial Differ. Eq. (2020), 1–20. http://dx.doi.org/10.1002/num.22642.

[47] M. Mohan Raja, and V. Vijayakumar, *New results concerning to approximate controllability of fractional integro-differential evolution equations of order $1 < \mathrm{r} < 2$*, Numer. Methods Partial Differ. Eq. (2020), 1–16. http://dx.doi.org/10.1002/num.22653.

[48] M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, and K. S. Nisar, *Results on existence and controllability results for fractional evolution inclusions of order* $1<r<2$ *with Clarke's subdifferential type*, Numer. Methods Partial Differ. Eq. (2020), 1–20. http://dx.doi.org/10.1002/num.22691.

[49] V. Vijayakumar, R. Udhayakumar, S. K. Panda, and K. S. Nisar, *Results on approximate controllability of Sobolev type fractional stochastic evolution hemivariational inequalities*, Numer. Methods Partial Differ. Eq. (2020), 1–20. http://dx.doi.org/10.1002/num.22690.

[50] W. W. Kavitha, V. Vijayakumar, R. Udhayakumar, S. K. Panda, and K. S. Nisar, *Existence and controllability of nonlocal mixed Volterra-Fredholm type fractional delay integro-differential equations of order* $1<r<2$, Numer. Methods Partial Differ. Eq. (2020), 1–21. http://dx.doi.org/10.1002/num.22697.

How to cite this article: Vijayakumar V, Ravichandran C, Nisar KS, Kucche KD. New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$. *Numer Methods Partial Differential Eq.* 2021;1–19. https://doi.org/10.1002/num.22772