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Research Article

b-Chromatic Number of Triple Star Graph Families

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Abstract. A b-coloring of a graph G is a proper coloring of the vertices of G such that there exists a vertex in each color class joined to atleast a vertex in each other color class, such a vertex is called a dominating vertex. The b-chromatic number of a graph G, denoted by $b(G)$, is the maximal integer k such that G may have a b-coloring by k colors. In this paper, we investigate the b-chromatic number of Central graph, Middle graph, Total graph and Line graph of Triple Star graph, denoted by $C(K_{1,n,n,n})$, $M(K_{1,n,n,n})$, $T(K_{1,n,n,n})$ and $L(K_{1,n,n,n})$, respectively.

Keywords. Central graph; Middle graph; Total graph; Line graph; Star graph; b-coloring; b-chromatic number

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1. Introduction

Let G be a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$ having no loops and multiple edges. All graphs considered here are undirected. In this paper, the term coloring will be used to define vertex coloring of graphs. A proper coloring of a graph G is the coloring of the vertices of G such that no two neighbors in G are assigned the same color. This paper deals with the b-chromatic number of graphs derived by several different Constructions from a Triple star graph.

The b-chromatic number $\varphi(G)$ [5, 7] of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class i contains atleast one vertex in each of the other color classes. Such a coloring is called a b-coloring. This concept of b-chromatic number was introduced in 1999 by Irving and Manlove [5], who proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees. Effantin and Kheddouci studied [2–4] the b-chromatic number for the powers of Path, Cycle, Complete Binary Tree, and Complete Caterpillar.

It has been proved in [6] by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\varphi(G) = d + 1$. Recently, motivated by the works of Sandi Klavžar and Marko Jakovac [7], who proved that the b-chromatic number of cubic graphs is 4 expect for the Petersen graph, $K_{3,3}$, the prism over K_3, and one more sporadic example with 10 vertices.

2. Preliminaries

Definition 2.1. The central graph $C(G)$ of a graph is obtained by subdividing each edge of G exactly once and joining all the non adjacent vertices of G.

Definition 2.2. The middle graph of G, denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G)E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

(a) x, y are in $E(G)$ and x, y are adjacent in G.

(b) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Definition 2.3. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of G is denoted by $T(G)$ and is defined as follows.

The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ is adjacent in $T(G)$, if one of the following holds:

(a) x, y are in $V(G)$ and x is adjacent to y in G.

(b) x, y are in $E(G)$ and x, y are adjacent in G.

(c) x is in $V(G)$, y is in $E(G)$ and x, y are adjacent in G.

Definition 2.4. Triple star $K_{1,n,n,n}$ is a tree obtained from the double star $K_{1,n,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $3n+1$ vertices and $3n$ edges.

3. *b*-Chromatic Number of Central Graph of Triple Star Graph

Algorithm 3.1.

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning b-coloring to the vertices of $C\ (K_{1,n,n,n})$.

begin

for $i = 1$ *to* n

{

$V_1 = \{p_i\}$;

$C(p_i) = i+1$;

}

{

$V_2 = \{m_i\}$;

$C(m_i) = n+1+i$;

$V_3 = \{y_i\}$;

$C(y_i) = 1$;

$V_4 = \{z_i\}$;

$C(z_i) = 1$;

$V_5 = \{q_i\}$;

$C(q_i) = i+1$;

$V_6 = \{x_i\}$;

$C(x_i) = i+2$;

}

$V_7 = \{v\}$;

$C(v) = 1$;

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7$.

end.

Theorem 3.1. *For a triple star graph $(K_{1,n,n,n})$, $n \geq 1$, the b chromatic number of the Central Graph $C(K_{1,n,n,n})$ is given by:*

$$\varphi(C(K_{1,n,n,n})) = 2n + 1.$$

Proof. By the definition of Central graph, the Central Graph $C(G)$ of the graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G. Let the edge vp_i, p_iq_i and q_im_i $(1 \leq i \leq n)$ be subdivided by the vertices x_i $(1 \leq i \leq n)$, y_i $(1 \leq i \leq n)$ and z_i $(1 \leq i \leq n)$ in $C(K_{1,n,n,n})$, respectively.

Clearly,

$$V(C(K_{1,n,n,n})) = \{v\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{m_i : 1 \leq i \leq n\}$$
$$\cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}.$$

The vertices $\{p_i : 1 \leq i \leq n\}$ induce a clique of order n (say K_n) and for $1 \leq i \leq n$ the vertices v, q_i and m_i induce a clique of order $n+1$ (say K_{n+1}) in $C(K_{1,n,n,n})$, respectively.

Now consider the vertex set $V(C(K_{1,n,n,n}))$ and the color class
$C = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}, c_{2n+1}\}$.
Assign a proper coloring to $C(K_{1,n,n,n})$ by Algorithm 3.1.

Thus we have, $\varphi(C(K_{1,n,n,n})) \geq 2n+1$.

Let us assume that $\varphi(C(K_{1,n,n,n})) > 2n+1$.

Suppose $\varphi(C(K_{1,n,n,n})) = 2n+2$. Since $\deg(x_i) = \deg(y_i) = \deg(z_i) = 2$.

The only possibility is to assign the color c_{2n+2} to the vertex set $\{p_i : 1 \leq i \leq n\}$ and $\{q_i : 1 \leq i \leq n\}$. But, if we assign the color c_{2n+2} to any vertex of $\{p_i : 1 \leq i \leq n\}$ and $\{q_i : 1 \leq i \leq n\}$, an easy check shows that, it will not produce a b-coloring.

Which is a contradiction. Therefore, assigning $2n+2$ colors is impossible.

Thus, we have, $\varphi(C(K_{1,n,n,n})) \leq 2n+1$. Hence $\varphi(C(K_{1,n,n,n})) = 2n+1$. □

4. *b*-Chromatic Number of Middle Graph of Triple Star Graph

Algorithm 4.1.

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning b-coloring to the vertices of $M\ (K_{1,n,n,n})$.

begin

for $i = 1$ *to* n

{

$V_1 = \{x_i\}$;

$C(x_i) = i$;

}

$V_2 = \{v\}$;

$C(v) = n+1$;

for $i = 1$ *to* n

{

$V_3 = \{p_i\}$;

$C(p_i) = n+1$;

$V_4 = \{q_i\}$;

$C(q_i) = n+1$;

$V_5 = \{m_i\}$;

$C(m_i) = n+1$;

}

for $i = 1$ *to* $n-1$

{

$V_6 = \{y_i\}$;

$C(y_i) = n;$

}

$C(y_n) = 1;$

for $i = 1$ *to* $n-1$

{

$V_7 = \{z_i\};$

$C(z_i) = 1;$

}

$C(z_n) = n;$

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7;$

end.

Theorem 4.1. *For a triple star graph* $(K_{1,n,n,n})$*,* $n \geq 4$*, the b chromatic number of the Middle Graph* $M(K_{1,n,n,n})$ *is given by:*

$$\varphi(M(K_{1,n,n,n})) = n+1.$$

Proof. By the definition of Middle graph, each edge vp_i, p_iq_i and q_im_i $(1 \leq i \leq n)$ in $(K_{1,n,n,n})$ are subdivided by the vertices x_i, y_i and z_i in $M(K_{1,n,n,n})$. The vertex set of Middle graph of Triple star graph is defined as,

$$V(M(K_{1,n,n,n})) = \{v\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{m_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}.$$

The vertices v, $x_1, x_2, \ldots, x_n$ induce a clique of order $n+1$ (say K_{n+1}) in $M(K_{1,n,n,n})$.

Now consider the vertex set $V(M(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}\}$.

Assign a proper coloring to $M(K_{1,n,n,n})$ by Algorithm 4.1.

Thus we have, $\varphi(M(K_{1,n,n,n})) \geq n+1$.

Let us assume that $\varphi(M(K_{1,n,n,n})) > n+1$.

Suppose, $\varphi(M(K_{1,n,n,n})) = n+2$, there must be atleast $n+2$ vertices of degree $n+1$ in $M(K_{1,n,n,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But, then these must be the vertices $\{v, x_1, x_2, \ldots, x_n\}$, since these are only the vertices with degree atleast $n+1$. Which is the contradiction. Therefore, $n+2$ colors is impossible. Thus, we have, $\varphi(M(K_{1,n,n,n})) \le n+1$. Hence $\varphi(M(K_{1,n,n,n})) = n+1$. □

Remark 4.1. For any positive integer n for $1 \le n \le 3$, $\varphi(M(K_{1,n,n,n})) = n+2$.

5. *b*-Chromatic Number of Total Graph of Triple Star Graph

Algorithm 5.1.

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning b-coloring to the vertices of $T(K_{1,n,n,n})$.

begin

for $i = 1$ *to* n

{

$V_1 = \{x_i\}$;

$C(x_i) = i+1$;

}

$V_2 = \{v\}$;

$C(v) = 1$;

for $i = 1$ *to* $n-1$

{

$V_3 = \{p_i\}$;

$C(p_i) = i+2$;

}

$C(p_n) = 2$;

for $i = 1$ *to* n

{

$V_4 = \{y_i\}$;

$C(y_i) = i;$

$V_5 = \{m_i\};$

$C(m_i) = i;$

$V_6 = \{q_i\};$

$C(q_i) = i + 1;$

}

for $i = 1$ *to* $n - 1$

{

$V_7 = \{z_i\};$

$C(z_i) = i + 2;$

}

$C(z_n) = 2;$

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7;$

end.

Theorem 5.1. *For a triple star graph* $(K_{1,n,n,n})$*,* $n \geq 4$*, the b chromatic number of the Total Graph* $T(K_{1,n,n,n})$ *is given by:*

$$\varphi(T(K_{1,n,n,n})) = n + 1.$$

Proof. By the definition of Total graph, each edge vp_i, p_iq_i and q_im_i $(1 \leq i \leq n)$ in $(K_{1,n,n,n})$ are subdivided by the vertices x_i, y_i and z_i in $T(K_{1,n,n,n})$. The vertex set of Total graph of Triple star graph is defined as,

$$V(T(K_{1,n,n,n})) = \{v\} \cup \{p_i : 1 \leq i \leq n\} \cup \{q_i : 1 \leq i \leq n\} \cup \{m_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}.$$

The vertices v, $x_1, x_2, \ldots, x_n$ induce a clique of order $n + 1$ (say K_{n+1}) in $T(K_{1,n,n,n})$.

Now consider the vertex set $V(T(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}\}$.

Assign a proper coloring to $T(K_{1,n,n,n})$ by Algorithm 5.1.

Thus we have, $\varphi(T(K_{1,n,n,n})) \geq n + 1$.

Let us assume that $\varphi(T(K_{1,n,n,n})) > n+1$. Suppose, $\varphi(T(K_{1,n,n,n})) = n+2$, there must be atleast $n+2$ vertices of degree $n+1$ in $T(K_{1,n,n,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But, then these must be the vertices $\{v, x_1, x_2, \ldots, x_n\}$, since these are only the vertices with degree at least $n+1$. Which is the contradiction. Therefore, $n+2$ colors is impossible. Thus, we have, $\varphi(T(K_{1,n,n,n})) \le n+1$. Hence $\varphi(T(K_{1,n,n,n})) = n+1$. □

6. *b*-Chromatic Number of Line Graph of Triple Star Graph

Algorithm 6.1.

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning b-coloring to the vertices of $L(K_{1,n,n,n})$.

begin

for $i = 1$ *to* n

{

$V_1 = \{m_i\}$;

$C(m_i) = i$;

}

{

$V_2 = \{q_i\}$;

If i = odd;

$C(q_i) = 2$;

If i = even;

$C(q_i) = 3$;

}

{

$V_3 = \{p_i\}$;

$C(p_i) = i$;

}

$V = V_1 \cup V_2 \cup V_3$;

end.

Theorem 6.1. *For a triple star graph* $(K_{1,n,n,n})$, $n \geq 3$, *the b chromatic number of the Line Graph* $L(K_{1,n,n,n})$ *is given by:*

$$\varphi(L(K_{1,n,n,n})) = n.$$

Proof. By the definition of Line graph, each edge of $(K_{1,n,n,n})$ taken to be as vertex in $L(K_{1,n,n,n})$. The vertex set of Line graph of Triple star graph is defined as,

$$V(L(K_{1,n,n,n})) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}.$$

The vertices $\{x_1, x_2, \ldots, x_n\}$ induce a clique of order n in $L(K_{1,n,n,n})$ (say K_n).

Now consider the vertex set $V(L(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, c_3, \ldots, c_n\}$.

Assign a proper coloring to $L(k_{1,n,n,n})$ by Algorithm 6.1.

Thus we have, $\varphi(L(K_{1,n,n,n})) \geq n$.

Let us assume that $\varphi(L(K_{1,n,n,n})) > n$. Suppose, $\varphi(L(K_{1,n,n,n})) = n+1$, there must be atleast $n+1$ vertices of degree n in $L(K_{1,n,n,n})$, all with distinct colors, and each adjacent to vertices of all of the other colors. But, then these must be the vertices $\{x_1, x_2, \ldots, x_n\}$, since these are only the vertices with degree at least n. Which is the contradiction. Therefore, $n+2$ colors is impossible. Thus, we have, $\varphi(L(K_{1,n,n,n})) \leq n$. Hence $\varphi(L(K_{1,n,n,n})) = n$. □

Remark 6.1. For any positive integer n $(1 \leq n \leq 2)$, $\varphi(L(K_{1,n,n,n})) = n+1$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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